COSMOLOGICAL IMPRINT OF QUANTUM VACUUM FLUCTUATIONS

Emilio Elizalde

ICE/CSIC & IEEC, UAB, Barcelona

Spanish Relativity Meeting, ERE, Tenerife, 10-14 Sep, 2007
Outline of this presentation
(soft intro)

- Observation: Universe expansion accelerates
Outline of this presentation
(soft intro)

- Observation: Universe expansion accelerates
- Explanation:
Outline of this presentation (soft intro)

- **Observation:** Universe expansion accelerates
- **Explanation:**
  - Within Einsteinian Gravity:
    - Quantum Vacuum Fluctuations (CC)
Outline of this presentation
(soft intro)

- Observation: Universe expansion accelerates
- Explanation:
  - Within Einsteinian Gravity: Quantum Vacuum Fluctuations (CC)
  - Modify Einsteinian Gravity: Tensor, Scalar-Tensor, Phantom, ...
Outline of this presentation
(soft intro)

- Observation: Universe expansion accelerates
- Explanation:
  - Within Einstenian Gravity: Quantum Vacuum Fluctuations (CC)
  - Modify Einstenian Gravity: Tensor, Scalar-Tensor, Phantom, ...
- Problems:
Outline of this presentation (soft intro)

- **Observation:** Universe expansion accelerates

- **Explanation:**
  - Within Einsteinian Gravity: Quantum Vacuum Fluctuations (CC)
  - Modify Einsteinian Gravity: Tensor, Scalar-Tensor, Phantom, ...

- **Problems:**
  - Regularization, QFT with Boundaries (technical, fundamental)
Outline of this presentation (soft intro)

- Observation: Universe expansion accelerates
- Explanation:
  - Within Einsteinnian Gravity: Quantum Vacuum Fluctuations (CC)
  - Modify Einsteinnian Gravity: Tensor, Scalar-Tensor, Phantom, ...
- Problems:
  - Regularization, QFT with Boundaries (technical, fundamental)
  - Observational restrictions (CC problem, SS tests, cosmological parameters)
Outline of this presentation
(soft intro)

- Observation: Universe expansion accelerates
- Explanation:
  - Within Einsteinian Gravity: Quantum Vacuum Fluctuations (CC)
  - Modify Einsteinian Gravity: Tensor, Scalar-Tensor, Phantom, ...
- Problems:
  - Regularization, QFT with Boundaries (technical, fundamental)
  - Observational restrictions (CC problem, SS tests, cosmological parameters)
- Here only: regularization (sound mathematics), Casimir (experiments), CC (ideas)
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of

3. $E$, $n$-dim vector bundle over

4. $M$ closed $n$-dim manifold
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of $E$, $n$-dim vector bundle over $M$

3. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr} A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of $A$, $s_0 = \dim M/\text{ord } A$ abscissa of converg of $\zeta_A(s)$
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of

3. $E$, $n$-dim vector bundle over

4. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr} A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of $A$, $s_0 = \dim M/\text{ord } A$ abscissa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of $E$, $n$-dim vector bundle over $M$

3. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:
$$
\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0
$$

$\{\lambda_j\}$ ordered spec of $A$, $s_0 = \dim M/\text{ord } A$ abscissa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$
Existence of $\zeta_A$ for $A$ a $\Psi DO$

1. $A$ a positive-definite elliptic $\Psi DO$ of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of $E$, $n$-dim vector bundle over $M$

3. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of $A$, $s_0 = \dim M/\text{ord } A$ abscissa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$

(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \quad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$
Definition of Determinant

\[ H \Psi \text{DO operator} \{ \varphi_i, \lambda_i \} \text{ spectral decomposition} \]
Definition of Determinant

\( H \Psi \) DO operator \( \{ \varphi_i, \lambda_i \} \) spectral decomposition

\[ \prod_{i \in I} \lambda_i \]

\[ \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]
Definition of Determinant

\( H \) \( \Psi \) DO operator \( \{ \varphi_i, \lambda_i \} \) spectral decomposition

\[ \prod_{i \in I} \lambda_i \quad \text{?} \]

\[ \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]

Riemann zeta func: \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re} \; s > 1 \) (\& analytic cont)

Definition: zeta function of \( H \)

\[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \; H^{-s} \]

As Mellin transform: \( \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \; t^{s-1} \text{tr} \; e^{-tH}, \quad \text{Re} \; s > s_0 \)

Derivative: \[ \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \]
Definition of Determinant

\[ H \] \( \psi \) DO operator \( \{ \varphi_i, \lambda_i \} \) spectral decomposition

\[ \prod_{i \in I} \lambda_i \quad \text{?!} \]

\[ \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]

Riemann zeta func: \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \ Re \ s > 1 \) (\& analytic cont)

Definition: zeta function of \( H \)

\[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s} \]

As Mellin transform: \( \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{tr} e^{-tH}, \ Re \ s > s_0 \)

Derivative: \( \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \)

Determinant: Ray & Singer, '67

\[ \det_\zeta H = \exp \left[ -\zeta'_H(0) \right] \]
**Definition of Determinant**

\[ H \] DO operator \( \{ \varphi_i, \lambda_i \} \) spectral decomposition

\[ \prod_{i \in I} \lambda_i \, ?? \]

\[ \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]

Riemann zeta func: \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \, \text{Re} \, s > 1 \) (& analytic cont)

Definition: zeta function of \( H \)

\[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \, H^{-s} \]

As Mellin transform: \( \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{tr} \, e^{-tH}, \, \text{Re} \, s > s_0 \)

Derivative: \( \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \)

Determinant: Ray & Singer, '67

\[ \det_\zeta H = \exp \left[ -\zeta'_H(0) \right] \]

Weierstrass def: subtract leading behavior of \( \lambda_i \) in \( i \), as \( i \to \infty \), until series \( \sum_{i \in I} \ln \lambda_i \) converges \( \implies \) non-local counterterms !!
Definition of Determinant

\( H \) \( \Psi \) DO operator \( \{ \varphi_i, \lambda_i \} \) spectral decomposition

\[
\prod_{i \in I} \lambda_i \quad \text{?!}
\]

\[
\ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i
\]

Riemann zeta func: \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re} \ s > 1 \) \( \text{(& analytic cont)} \)

Definition: zeta function of \( H \)

\( \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s} \)

As Mellin transform: \( \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{tr} \ e^{-tH}, \quad \text{Re} \ s > s_0 \)

Derivative: \( \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \)

Determinant: Ray & Singer, ’67

\[
\det_{\zeta} H = \exp \left[ -\zeta'_H(0) \right]
\]

Weierstrass def: subtract leading behavior of \( \lambda_i \) in \( i \), as \( i \to \infty \),

until series \( \sum_{i \in I} \ln \lambda_i \) converges \( \implies \) non-local counterterms !

C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...
Consequences of the Multipl Anomaly

In the path integral formulation

\[
\int [d\Phi] \exp \left\{ - \int d^D x \left[ \Phi^\dagger(x) \Phi(x) + \cdots \right] \right\}
\]

Gaussian integration: \[ \rightarrow \det ( \ ) ^\pm \]

\[
\begin{pmatrix}
  A_1 & A_2 \\
  A_3 & A_4 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
  A \\
  B \\
\end{pmatrix}
\]

\[
\det(AB) \quad \text{or} \quad \det A \cdot \det B \quad ?
\]
Consequences of the Multipl Anomaly

In the path integral formulation

\[
\int [d\Phi] \exp \left\{ - \int d^D x \left[ \Phi^\dagger(x)(\cdots)\Phi(x) + \cdots \right] \right\}
\]

Gaussian integration:

\[
\begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
A \\
B
\end{pmatrix}
\]

\[
\det(AB) \quad \text{or} \quad \det A \cdot \det B \quad ?
\]

In a situation where a superselection rule exists, \(AB\) has no sense (much less its determinant):

\[
\rightarrow \det A \cdot \det B
\]
Consequences of the Multipl Anomaly

In the path integral formulation

\[
\int [d\Phi] \exp \left\{ - \int d^D x \left[ \Phi^\dagger(x) (\cdots \Phi(x) + \cdots) \right] \right\}
\]

Gaussian integration:

\[
\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \rightarrow \begin{pmatrix} A \\ B \end{pmatrix}
\]

\[
\det(AB) \quad \text{or} \quad \det A \cdot \det B
\]

In a situation where a superselection rule exists, \( AB \) has no sense (much less its determinant):

\[
\Rightarrow \det A \cdot \det B
\]

But if diagonal form obtained after change of basis (diag. process), the preserved quantity is:

\[
\Rightarrow \det(AB)
\]
The Chowla-Selberg Expansion Formula: Basics

- **Jacobi’s identity** for the $\theta$–function

\[
\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2n z), \quad q := e^{i \pi \tau}, \tau \in \mathbb{C}
\]

\[
\theta_3(z, \tau) = \frac{1}{\sqrt{-i \tau}} e^{z^2 / i \pi \tau} \theta_3 \left( \frac{z}{\tau} \left| -\frac{1}{\tau} \right. \right)
\]

equivalently:

\[
\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2 \pi n z), \quad z, t \in \mathbb{C}, \ Re \ t > 0
\]
The Chowla-Selberg Expansion Formula: Basics

- **Jacobi’s identity** for the \( \theta \)–function

\[
\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^n \cos(2nz), \quad q := e^{i\pi \tau}, \quad \tau \in \mathbb{C}
\]

\[
\theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi \tau} \theta_3 \left( \frac{z}{\tau} \mid \frac{-1}{\tau} \right)
\]

equivalently:

\[
\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi nz), \quad z, t \in \mathbb{C}, \quad \Re t > 0
\]

- **Higher dimensions**: **Poisson summ formula** (Riemann)

\[
\sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \tilde{f}(\vec{m})
\]

\(\tilde{f}\) Fourier transform

[Gerbert + Miller, BAMS '03, Iwaniec, Morgan, ICM '06]
The Chowla-Selberg Expansion Formula: Basics

- **Jacobi’s identity** for the $\theta$–function

$$\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^n \cos(2nz), \quad q := e^{i\pi \tau}, \quad \tau \in \mathbb{C}$$

$$\theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi \tau} \theta_3 \left( \frac{z}{\tau} \bigg| \frac{-1}{\tau} \right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 \tau} = \sqrt{\frac{\pi}{\tau}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{\tau}} \cos(2\pi nz), \quad z, \tau \in \mathbb{C}, \quad \text{Re} \ t > 0$$

- **Higher dimensions:** Poisson summ formula (Riemann)

$$\sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \tilde{f}(\vec{m})$$

$\tilde{f}$ Fourier transform

[Gelbart + Miller, BAMS ’03, Iwaniec, Morgan, ICM ’06]

- **Truncated sums** → asymptotic series
Extended CS Formulas (ECS)

Consider the zeta function \( \Re s > p/2, A > 0, \Re q > 0 \)

\[
\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{n} \in \mathbb{Z}^p} [Q (\vec{n} + \vec{c}) + q]^{-s}
\]

\text{prime: point } \vec{n} = \vec{0} \text{ to be excluded from the sum}

(inescapable condition when } c_1 = \cdots = c_p = q = 0 )

\[
Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}
\]
Extended CS Formulas (ECS)

Consider the zeta function \((\text{Re} \, s > p/2, A > 0, \text{Re} \, q > 0)\)

\[
\zeta_{A, \vec{c}, q}(s) = \sum_{\vec{n} \in Z^p} \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{n} \in Z^p} [Q (\vec{n} + \vec{c}) + q]^{-s} 
\]

prime: point \(\vec{n} = \vec{0}\) to be excluded from the sum

(inescapable condition when \(c_1 = \cdots = c_p = q = 0\))

\[
Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}
\]

Case \(q \neq 0\) \((\text{Re} \, q > 0)\)

\[
\zeta_{A, \vec{c}, q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2} \pi^s q^{-s/2+p/4}}{\sqrt{\det A} \, \Gamma(s)} 
\]

\[
\times \sum_{\vec{m} \in Z^p_{1/2}}' \cos(2\pi \vec{m} \cdot \vec{c}) \left( \vec{m}^T A^{-1} \vec{m} \right)^{s/2-p/4} K_{p/2-s} \left( 2\pi \sqrt{2q \vec{m}^T A^{-1} \vec{m}} \right)
\]

[ECS1]
Extended CS Formulas (ECS)

Consider the zeta function \((\text{Re } s > p/2, A > 0, \text{Re } q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} ' \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{n} \in \mathbb{Z}^p} ' \left[ Q (\vec{n} + \vec{c}) + q \right]^{-s}
\]

prime: point \(\vec{n} = \vec{0}\) to be excluded from the sum
(inescapable condition when \(c_1 = \cdots = c_p = q = 0\))

\[
Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}
\]

Case \(q \neq 0 (\text{Re } q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2} \pi^{s} q^{-s/2+p/4}}{\sqrt{\det A} \Gamma(s)}
\]

\[
\times \sum_{\vec{m} \in \mathbb{Z}_{1/2}^p} ' \cos(2\pi \vec{m} \cdot \vec{c}) \left( \vec{m}^T A^{-1} \vec{m} \right)^{s/2-p/4} K_{p/2-s} \left( 2\pi \sqrt{2q} \vec{m}^T A^{-1} \vec{m} \right)
\]

[Pole: \(s = p/2\)  Residue:]

\[
\text{Res}_{s=p/2} \zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}
\]

ERE, Tenerife, September 13, 2007 – p. 7/17
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.

Exhibits singularities (simple poles) of the meromorphic continuation —with the corresponding residua— explicitly.
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane

Exhibits singularities (simple poles) of the meromorphic continuation—with the corresponding residua—explicitly

Only condition on matrix $A$: corresponds to (non negative) quadratic form, $Q$. Vector $\vec{c}$ arbitrary, while $q$ is (to start) a non-neg constant
1. Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.
2. Exhibits singularities (simple poles) of the meromorphic continuation—with the corresponding residua—explicitly.
3. Only condition on matrix $A$: corresponds to (non negative) quadratic form, $Q$. Vector $\vec{c}$ arbitrary, while $q$ is (to start) a non-neg constant.
4. $K_\nu$ modified Bessel function of the second kind and the subindex $1/2$ in $\mathbb{Z}_{1/2}^p$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose first non-zero component is positive].
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.

Exhibits singularities (simple poles) of the meromorphic continuation—with the corresponding residua—explicitly.

Only condition on matrix $A$: corresponds to (non negative) quadratic form, $Q$. Vector $\vec{c}$ arbitrary, while $q$ is (to start) a non-neg constant.

$K_\nu$ modified Bessel function of the second kind and the subindex $1/2$ in $\mathbb{Z}^{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$.

[simple criterion: one may select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose first non-zero component is positive]

Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[ \pi^{j/2} a_{p-j}^{j/2-s} \Gamma \left( s - \frac{j}{2} \right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{j-s/2} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j} n^{j/2-s} (\vec{m}_j^t A_j^{-1} \vec{m}_j)^{s/2-j/4} K_{j/2-s} \left( 2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j} \right) \right]$$
Zero point energy

QFT vacuum to vacuum transition: $\langle 0 | H | 0 \rangle$
Zero point energy

QFT vacuum to vacuum transition: \[ \langle 0 | H | 0 \rangle \]

Spectrum, normal ordering (harm oscill):

\[ H = \left( n + \frac{1}{2} \right) \lambda_n \ a_n \ a_n^\dagger \]
Zero point energy

QFT vacuum to vacuum transition: \[ \langle 0 | H | 0 \rangle \]

Spectrum, normal ordering (harm oscill):

\[ H = \left( n + \frac{1}{2} \right) \lambda_n a_n a_n^\dagger \]

\[ \langle 0 | H | 0 \rangle = \frac{\hbar c}{2} \sum_n \lambda_n = \frac{1}{2} \text{tr} \ H \]
Zero point energy

QFT vacuum to vacuum transition: \( \langle 0 | H | 0 \rangle \)

Spectrum, normal ordering (harm oscill):

\[
H = \left( n + \frac{1}{2} \right) \lambda_n \ a_n \ a_n^\dagger
\]

\[
\langle 0 | H | 0 \rangle = \frac{\hbar c}{2} \sum_n \lambda_n = \frac{1}{2} \ \text{tr} \ H
\]

gives \( \infty \) physical meaning?
Zero point energy

QFT vacuum to vacuum transition: \( \langle 0 | H | 0 \rangle \)

Spectrum, normal ordering (harm oscill):

\[
H = \left( n + \frac{1}{2} \right) \lambda_n \ a_n \ a_n^\dagger
\]

\[
\langle 0 | H | 0 \rangle = \frac{\hbar c}{2} \sum_n \lambda_n = \frac{1}{2} \text{tr} \ H
\]

gives \( \infty \) physical meaning?

Regularization + Renormalization (cut-off, dim, \( \zeta \))
Zero point energy

QFT vacuum to vacuum transition: \( \langle 0 | H | 0 \rangle \)

Spectrum, normal ordering (harm oscill):

\[
H = \left( n + \frac{1}{2} \right) \lambda_n \ a_n \ a_n^\dagger
\]

\[
\langle 0 | H | 0 \rangle = \frac{\hbar c}{2} \sum_n \lambda_n = \frac{1}{2} \text{tr} \ H
\]

gives \( \infty \) physical meaning?

Regularization + Renormalization (cut-off, dim, \( \zeta \))

Even then: Has the final value real sense?
The Casimir Effect
The Casimir Effect

BC e.g. periodic

Casimir Effect

vacuum
The Casimir Effect

BC  e.g. periodic
⇒ all kind of fields

Casimir Effect
The Casimir Effect

BC  e.g. periodic
⇒  all kind of fields
⇒  curvature or topology
The Casimir Effect

BC e.g. periodic
⇒ all kind of fields
⇒ curvature or topology

Universal process:
The Casimir Effect

BC  e.g. periodic

⇒ all kind of fields

⇒ curvature or topology

Universal process:

- Sonoluminiscence  (Schwinger)
- Cond. matter  (wetting $^3$He alc.)
- Optical cavities
- Direct experim. confirmation
The Casimir Effect

BC  e.g. periodic
⇒  all kind of fields
⇒  curvature or topology

Universal process:

- Sonoluminiscence  \(^{(Schwinger)}\)
- Cond. matter  \(^{(wetting \, ^3\text{He alc.})}\)
- Optical cavities
- Direct experim. confirmation

Van der Waals, Lifschitz theory
The Casimir Effect

BC e.g. periodic
⇒ all kind of fields
⇒ curvature or topology

Universal process:
- Sonoluminiscence (Schwinger)
- Cond. matter (wetting $^3$He alc.)
- Optical cavities
- Direct experim. confirmation

Van der Waals, Lifschitz theory

Dynamical CE ⇐
Lateral CE
Extract energy from vacuum
CE and the cosmological constant ⇐
The standard approach
The standard approach

Casimir force: calculated by computing change in zero point energy of the em field
The standard approach

Casimir force: calculated by computing change in zero point energy of the EM field

But Casimir effects can be calculated as $S$-matrix elements: Feynman diagrams with ext. lines
The standard approach

\[ \text{Casimir force: calculated by computing change in zero point energy of the em field} \]

\[ \text{But Casimir effects can be calculated as } S\text{-matrix elements:} \]

\[ \text{Feynman diagrams with ext. lines} \]

In modern language the Casimir energy can be expressed in terms of the trace of the Greens function for the fluctuating field in the background of interest (conducting plates)

\[ \mathcal{E} = \frac{\hbar}{2\pi} \text{Im} \int d\omega \omega \text{Tr} \int d^3x \left[ G(x, x, \omega + i\epsilon) - G_0(x, x, \omega + i\epsilon) \right] \]
The standard approach

Casimir force: calculated by computing change in zero point energy of the em field

But Casimir effects can be calculated as $S$-matrix elements: Feynman diagrams with ext. lines

In modern language the Casimir energy can be expressed in terms of the trace of the Greens function for the fluctuating field in the background of interest (conducting plates)

$$\mathcal{E} = \frac{\hbar}{2\pi} \text{Im} \int d\omega \text{ Tr } \int d^3x [G(x, x, \omega + i\epsilon) - G_0(x, x, \omega + i\epsilon)]$$

$G$ full Greens function for the fluctuating field

$G_0$ free Greens function

Trace is over spin
$$E_C = \left\langle \text{plates} \right\rangle - \left\langle \text{no plates} \right\rangle$$
\[ E_C = \langle \text{plates} \rangle - \langle \text{no plates} \rangle \]

\[
\frac{1}{\pi} \text{Im} \int [G(x, x, \omega + i\epsilon) - G_0(x, x, \omega + i\epsilon)] = \frac{d\Delta N}{d\omega}
\]

change in the density of states due to the background
\[ E_C = \langle \text{plates} \rangle - \langle \text{no plates} \rangle \]

\[
\frac{1}{\pi} \text{Im} \int \left[ G(x, x, \omega + i\epsilon) - G_0(x, x, \omega + i\epsilon) \right] = \frac{d\Delta N}{d\omega}
\]

change in the density of states due to the background

\[ \implies \text{A restatement of the Casimir sum over shifts in zero-point energies} \]

\[ \frac{\hbar}{2} \sum (\omega - \omega_0) \]
$$E_C = \left\langle \text{plates} \right\rangle - \left\langle \text{no plates} \right\rangle$$

$$\frac{1}{\pi} \text{Im} \int [G(x, x, \omega + i\epsilon) - G_0(x, x, \omega + i\epsilon)] = \frac{d\Delta N}{d\omega}$$

change in the density of states due to the background

\[\Rightarrow \] A restatement of the Casimir sum over shifts in zero-point energies

\[\frac{\hbar}{2} \sum (\omega - \omega_0)\]

\[\Rightarrow \] Lippman-Schwinger eq. allows full Greens f, $G$, be expanded as a series in free Green's f, $G_0$, and the coupling to the external field
\[ E_C = \left\langle \text{plates} \right\rangle - \left\langle \text{no plates} \right\rangle \]

\[ \frac{1}{\pi} \text{Im} \int [G(x, x, \omega + i\epsilon) - G_0(x, x, \omega + i\epsilon)] = \frac{d\Delta N}{d\omega} \]

change in the density of states due to the background

\[ \Rightarrow \text{A restatement of the Casimir sum over shifts in zero-point energies} \]

\[ \frac{\hbar}{2} \sum (\omega - \omega_0) \]

\[ \Rightarrow \text{Lippman-Schwinger eq. allows full Greens f, } G, \text{ be expanded as a series in free Green’s f, } G_0, \text{ and the coupling to the external field} \]

\[ \Rightarrow \text{“Experimental confirmation of the Casimir effect does not establish the reality of zero point fluctuations” [R. Jaffe et. al.]} \]
The Dynamical Casimir Effect

The Dynamical Casimir Effect

- Moving mirrors modify structure of quantum vacuum
- Creation and annihilation of photons; once mirrors return to rest, some produced photons may still remain: flux of radiated particles
The Dynamical Casimir Effect


- Moving mirrors modify structure of quantum vacuum
- Creation and annihilation of photons; once mirrors return to rest, some produced photons may still remain: flux of radiated particles
- For a single, perfectly reflecting mirror:
  # photons & energy diverge while mirror moves
The Dynamical Casimir Effect


- Moving mirrors modify structure of quantum vacuum

- Creation and annihilation of photons; once mirrors return to rest, some produced photons may still remain: flux of radiated particles

- For a single, perfectly reflecting mirror:
  # photons & energy diverge while mirror moves

- Several renormalization prescriptions have been used in order to obtain a well-defined energy
The Dynamical Casimir Effect


- Moving mirrors modify structure of quantum vacuum

- Creation and annihilation of photons; once mirrors return to rest, some produced photons may still remain: flux of radiated particles

- For a single, perfectly reflecting mirror:
  # photons & energy diverge while mirror moves

- Several renormalization prescriptions have been used in order to obtain a well-defined energy

- Problem: for some trajectories this finite energy is not a positive quantity and cannot be identified with the energy of the photons
The Dynamical Casimir Effect


- Moving mirrors modify structure of quantum vacuum

- Creation and annihilation of photons; once mirrors return to rest, some produced photons may still remain: flux of radiated particles

- For a single, perfectly reflecting mirror:
  # photons & energy diverge while mirror moves

- Several renormalization prescriptions have been used in order to obtain a well-defined energy

- Problem: for some trajectories this finite energy is not a positive quantity and cannot be identified with the energy of the photons

  Moore; Razavy, Terning; Johnston, Sarkar; Dodonov et al; Plunien et al; Barton, Eberlein, Calogeracos; Ford, Vilenkin; Jaeckel, Reynaud, Lambrecht; Brevik, Milton et al; Dalvit, Maia-Neto et al; Law; Parentani, ...
A CONSISTENT APPROACH:

J. Haro & E.E., PRL 97 (2006); arXiv:0705.0597
A CONSISTENT APPROACH:
J. Haro & E.E., PRL 97 (2006); arXiv:0705.0597

- Partially transmitting mirrors, which become transparent to very high frequencies (analytic matrix)
- Proper use of a Hamiltonian method & corresponding renormalization
A CONSISTENT APPROACH:
J. Haro & E.E., PRL 97 (2006); arXiv:0705.0597

- Partially transmitting mirrors, which become transparent to very high frequencies (analytic matrix)
- Proper use of a Hamiltonian method & corresponding renormalization
- Proved both: # of created particles is finite & their energy is always positive, for the whole trajectory during the mirrors’ displacement
A CONSISTENT APPROACH:
J. Haro & E.E., PRL 97 (2006); arXiv:0705.0597

- Partially transmitting mirrors, which become transparent to very high frequencies (analytic matrix)
- Proper use of a Hamiltonian method & corresponding renormalization
- Proved both: # of created particles is finite & their energy is always positive, for the whole trajectory during the mirrors’ displacement
- The radiation-reaction force acting on the mirrors owing to emission-absorption of particles is related with the field’s energy through the energy conservation law: energy of the field at any \( t \) equals (with opposite sign) the work performed by the reaction force up to time \( t \)
A CONSISTENT APPROACH:
J. Haro & E.E., PRL 97 (2006); arXiv:0705.0597

- Partially transmitting mirrors, which become transparent to very high frequencies (analytic matrix)
- Proper use of a Hamiltonian method & corresponding renormalization
- Proved both: # of created particles is finite & their energy is always positive, for the whole trajectory during the mirrors’ displacement
- The radiation-reaction force acting on the mirrors owing to emission-absorption of particles is related with the field’s energy through the energy conservation law: energy of the field at any \( t \) equals (with opposite sign) the work performed by the reaction force up to time \( t \)
- Such force is split into two parts: a dissipative force whose work equals minus the energy of the particles that remain & a reactive force vanishing when the mirrors return to rest
A CONSISTENT APPROACH:
J. Haro & E.E., PRL 97 (2006); arXiv:0705.0597

- Partially transmitting mirrors, which become transparent to very high frequencies (analytic matrix)
- Proper use of a Hamiltonian method & corresponding renormalization
- Proved both: # of created particles is finite & their energy is always positive, for the whole trajectory during the mirrors’ displacement
- The radiation-reaction force acting on the mirrors owing to emission-absorption of particles is related with the field’s energy through the energy conservation law: energy of the field at any $t$ equals (with opposite sign) the work performed by the reaction force up to time $t$
- Such force is split into two parts: a dissipative force whose work equals minus the energy of the particles that remain & a reactive force vanishing when the mirrors return to rest
- The dissipative part we obtain agrees with the other methods. But those have problems with the reactive part, which in general yields a non-positive energy

⇒ EXPERIMENT
Quantum Vacuum Fluct’s & the CC

The main issue: S.A. Fulling et. al., hep-th/070209v2

energy ALWAYS gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor

\[ \langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu} \]
The main issue: energy **ALWAYS** gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor

\[ \langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu} \]

Appears on the rhs of Einstein's equations:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G (\tilde{T}_{\mu\nu} - \mathcal{E} g_{\mu\nu}) \]

It affects **cosmology**: \( \tilde{T}_{\mu\nu} \) excitations above the vacuum
The main issue: energy ALWAYS gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor 

\[ \langle T_{\mu\nu} \rangle \equiv -E g_{\mu\nu} \]

Appears on the rhs of Einstein’s equations:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G (\tilde{T}_{\mu\nu} - E g_{\mu\nu}) \]

It affects cosmology: \( \tilde{T}_{\mu\nu} \) excitations above the vacuum

Equivalent to a cosmological constant \( \lambda = 8\pi G E \)
Quantum Vacuum Fluct’s & the CC

The main issue: 

S.A. Fulling et. al., hep-th/070209v2

energy ALWAYS gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor

\[ \langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu} \]

Appears on the rhs of Einstein’s equations:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G (\tilde{T}_{\mu\nu} - \mathcal{E} g_{\mu\nu}) \]

It affects cosmology: \( \tilde{T}_{\mu\nu} \) excitations above the vacuum

Equivalent to a cosmological constant \( \lambda = 8\pi G \mathcal{E} \)

Recent observations: M. Tegmark et al. [SDSS Collab.] PRD 2004

\[ \lambda = (2.14 \pm 0.13 \times 10^{-3} \text{ eV})^4 \sim 4.32 \times 10^{-9} \text{ erg/cm}^3 \]
Quantum Vacuum Fluct’s & the CC

- The main issue: S.A. Fulling et. al., hep-th/070209v2
- energy ALWAYS gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor
  \[ \langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu} \]
- Appears on the rhs of Einstein’s equations:
  \[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G (\tilde{T}_{\mu\nu} - \mathcal{E} g_{\mu\nu}) \]
- It affects cosmology: \( \tilde{T}_{\mu\nu} \) excitations above the vacuum
- Equivalent to a cosmological constant \( \lambda = 8\pi G \mathcal{E} \)
- Recent observations: M. Tegmark et al. [SDSS Collab.] PRD 2004
  \[ \lambda = (2.14 \pm 0.13 \times 10^{-3} \text{ eV})^4 \sim 4.32 \times 10^{-9} \text{ erg/cm}^3 \]
- Idea: zero point fluctuations can contribute to the cosmological constant Ya.B. Zeldovich ’68
Relativistic field: collection of harmonic oscill’s (scalar field)

\[ E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = k^2 + m^2/\hbar^2, \quad k = 2\pi/\lambda \]
Relativistic field: collection of harmonic oscill's (scalar field)

\[ E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = k^2 + m^2/\hbar^2, \quad k = 2\pi/\lambda \]

Evaluating in a box and putting a cut-off at maximum \( k_{\text{max}} \) corresp'ng to QFT physics (e.g., Planck energy)

\[ \rho \sim \frac{\hbar k_{\text{Planck}}^4}{16\pi^2} \sim 10^{123} \rho_{\text{obs}} \]

kind of a modern (and thick!) aether

R. Caldwell, S. Carroll, ...
Relativistic field: collection of harmonic oscill’s (scalar field)

\[ E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = \frac{k^2 + m^2}{\hbar^2}, \quad k = \frac{2\pi}{\lambda} \]

Evaluating in a box and putting a cut-off at maximum \( k_{\text{max}} \) corresp’ng to QFT physics (e.g., Planck energy)

\[ \rho \sim \frac{\hbar k_{\text{Planck}}^4}{16\pi^2} \sim 10^{123}\rho_{\text{obs}} \]

kind of a modern (and thick!) aether R. Caldwell, S. Carroll, ...

Observational tests see nothing (or very little) of it:

\[ \Rightarrow (\text{new}) \text{ cosmological constant problem} \]
Relativistic field: collection of harmonic oscill’s (scalar field)

\[ E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = k^2 + m^2/\hbar^2, \quad k = \frac{2\pi}{\lambda} \]

Evaluating in a box and putting a cut-off at maximum \( k_{max} \) corresp’ng to QFT physics (e.g., Planck energy)

\[ \rho \sim \frac{\hbar k_{Planck}^4}{16\pi^2} \sim 10^{123} \rho_{obs} \]

kind of a modern (and thick!) aether R. Caldwell, S. Carroll, ...

Observational tests see nothing (or very little) of it:

\[ \implies \text{(new) cosmological constant problem} \]

Very difficult to solve and we do not address this question directly

[Baum, Hawking, Coleman, Polchinsky, Weinberg,...]
Relativistic field: collection of harmonic oscill’s (scalar field)

\[ E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = k^2 + m^2/\hbar^2, \quad k = \frac{2\pi}{\lambda} \]

Evaluating in a box and putting a cut-off at maximum \( k_{max} \) corresp’ng to QFT physics (e.g., Planck energy)

\[ \rho \sim \frac{\hbar k_{Planck}^4}{16\pi^2} \sim 10^{123} \rho_{obs} \]

kind of a modern (and thick!) aether

R. Caldwell, S. Carroll, ...

**Observational tests** see nothing (or very little) of it:

\[ \implies \text{(new) cosmological constant problem} \]

Very difficult to solve and we do not address this question directly

[Baum, Hawking, Coleman, Polchinsky, Weinberg,...]

**What we do consider** —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:

\[ \implies \text{kind of cosmological Casimir effect} \]
Cosmo-Topological Casimir Effect

Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}\text{mm}$ dims, bulk vs brane Susy breaking scales
* T. Padmanabhan, gr-qc/0606061: Holographic Perspective, CC is an intg const, no response of gravity to changes in bulk vac energy dens
Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}$mm dims, bulk vs brane Susy breaking scales
* T. Padmanabhan, gr-qc/0606061: Holographic Perspective, CC is an intg const, no response of gravity to changes in bulk vac energy dens

We show (with different examples) that this value acquires the correct order of magnitude — corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:
Cosmo-Topological Casimir Effect

Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}$mm dims, bulk vs brane Susy breaking scales
* T. Padmanabhan, gr-qc/0606061: Holographic Perspective, CC is an intg const, no response of gravity to changes in bulk vac energy dens

We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:

(a) small and large compactified scales
Cosmo-Topological Casimir Effect

Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}$mm dims, bulk vs brane Susy breaking scales
* T. Padmanabhan, gr-qc/0606061: Holographic Perspective, CC is an intg const, no response of gravity to changes in bulk vac energy dens

We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:

(a) small and large compactified scales
(b) dS & AdS worldbranes
Cosmo-Topological Casimir Effect

Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs.

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}\text{mm}$ dims, bulk vs brane Susy breaking scales
* T. Padmanabhan, gr-qc/0606061: Holographic Perspective, CC is an intg const, no response of gravity to changes in bulk vac energy dens

We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe—in some reasonable models involving:

- (a) small and large compactified scales
- (b) dS & AdS worldbranes
- (c) supergraviton theories (discret dims, deconstr)