Vacuum Fluctuations & Cosmology

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4th Sakharov Conference, Moscow, May 21, 2009
Outline of this presentation

- Einstein’s Cosmological Constant
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- On the Casimir Effect & the $\zeta$ Function Method
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- Gravity Eqs as Eqs of State
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With THANKS to:
G. Cognola, J. Haro, S.D. Odintsoy, P.J. Silva, S. Zerbini, ...
Einstein’s Cosmological Constant

Our universe seems to be spatially flat and to possess a non-vanishing cosmological constant

- For cosmologists and general relativists: a great mistake (Einstein)
  \[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -(8\pi G/c^4)T_{\mu\nu} + \lambda g_{\mu\nu} \]

- For elementary particle physicists: a great embarrassment
  no way to get rid off (Coleman, Weinberg, Polchinski)

- The cc \( \Lambda \) is indeed a peculiar quantity
  - has to do with cosmology Einstein’s eqs., FRW universe
  - has to do with the local structure of elementary particle physics
    stress-energy density \( \mu \) of the vacuum

  \[ L_{cc} = \int d^4x \sqrt{-g} \mu^4 = \frac{1}{8\pi G} \int d^4x \sqrt{-g} \lambda \]

  In other words: two contributions on the same footing (Zel’dovich, 68)

  \[ \frac{\Lambda c^2}{8\pi G} + \frac{1}{Vol} \frac{\hbar c}{2} \sum_i \omega_i \]
Zero point energy

QFT vacuum to vacuum transition: $\langle 0 | H | 0 \rangle$
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Spectrum, normal ordering (harm oscill):

\[
H = \left( n + \frac{1}{2} \right) \lambda_n \ a_n \ a_n^\dagger
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gives \( \infty \) physical meaning?
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Regularization + Renormalization (cut-off, dim, \( \zeta \))
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Regularization \( + \) Renormalization (cut-off, dim, \( \zeta \))

Even then: Has the final value real sense?
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$
2. $A$ acts on the space of smooth sections of $\mathcal{E}$, $n$-dim vector bundle over $M$ closed $n$-dim manifold
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(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$

(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \quad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$
Definition of Determinant

\[ H \Psi \text{DO operator} \{ \varphi_i, \lambda_i \} \text{ spectral decomposition} \]
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Definition: zeta function of \( H \) \[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s} \]

As Mellin transform: \[ \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{tr} \ e^{-tH}, \quad \Re s > s_0 \]

Derivative: \[ \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \]
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Determinant: Ray & Singer, '67

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**Weierstrass def:** subtract leading behavior of \( \lambda_i \) in \( i \), as \( i \to \infty \), until series \( \sum_{i \in I} \ln \lambda_i \) converges  
\[ \implies \] non-local counterterms !!
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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...
Properties

The definition of the determinant $\text{det}_\zeta A$ only depends on the homotopy class of the cut.
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A zeta function (and corresponding determinant) with the same meromorphic structure in the complex $s$-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C}\backslash\mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik]
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- Asymptotic expansion for the heat kernel:

  \[
  \text{tr } e^{-tA} = \sum_{\lambda \in \text{Spec } A} e^{-t\lambda}
  \]

  \[
  \sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0
  \]

  \[
  \alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \text{Res}_{s=s_j} \zeta_A(s), \quad s_j \notin -\mathbb{N}
  \]

  \[
  \alpha_j(A) = \frac{(-1)^k}{k!} \left[ \text{PP } \zeta_A(-k) + \psi(k + 1) \text{Res}_{s=-k} \zeta_A(s) \right],
  \]

  \[
  \beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N} \setminus \{0\}
  \]

  \[
  \text{PP } \phi := \lim_{s \to p} \left[ \phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]
  \]
The Chowla-Selberg Expansion Formula: Basics

**Jacobi’s identity** for the $\theta$–function

$$\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^n \cos(2nz), \quad q := e^{i\pi \tau}, \quad \tau \in \mathbb{C}$$

$$\theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi \tau} \theta_3 \left( \frac{z}{\tau} \mid -\frac{1}{\tau} \right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi nz), \quad z, t \in \mathbb{C}, \quad \text{Re} t > 0$$
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- **Higher dimensions:** Poisson summ formula (Riemann)
  \[
  \sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \tilde{f}(\vec{m})
  \]
  \[
  \tilde{f} \quad \text{Fourier transform}
  \]

  [Gelbart + Miller, BAMS ’03, Iwaniec, Morgan, ICM ’06]
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$\tilde{f}$ Fourier transform

[Gelbart + Miller, BAMS ’03, Iwaniec, Morgan, ICM ’06]

- **Truncated sums** $\rightarrow$ asymptotic series
Consider the zeta function \((\Re s > p/2, A > 0, \Re q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \sum'_{\vec{n} \in \mathbb{Z}^p} \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum'_{\vec{n} \in \mathbb{Z}^p} \left[ Q (\vec{n} + \vec{c}) + q \right]^{-s}
\]

prime: point \(\vec{n} = \vec{0}\) to be excluded from the sum
(inescapable condition when \(c_1 = \cdots = c_p = q = 0\))

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Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}
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Extended CS Formulas (ECS)

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Case \(q \neq 0 (\text{Re} q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4} + 2 \pi^s q^{-s/2+p/4}}{\sqrt{\det A} \Gamma(s)}
\]

\[\times \sum_{\vec{m} \in \mathbb{Z}^p_{1/2}}' \cos(2\pi \vec{m} \cdot \vec{c}) (\vec{m}^T A^{-1} \vec{m})^{s/2-p/4} K_{p/2-s} \left(2\pi \sqrt{2q \vec{m}^T A^{-1} \vec{m}}\right) \]

[ECS1]
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\]

Pole: \( s = p/2 \) Residue:

\[
\text{Res}_{s=p/2} \zeta_{A, \vec{c}, q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}
\]
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[simple criterion: one may select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose first non-zero component is positive]
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[simple criterion: one may select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose first non-zero component is positive]

Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

\[
\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} \left( \det A_j \right)^{-1/2} \left[ \pi^{j/2} a_{p-j}^{j/2-s} \Gamma \left( s - \frac{j}{2} \right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{j-s} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j} \langle \vec{m}_j A_j^{-1} \vec{m}_j \rangle^{s/2-j/4} K_{j/2-s} \left( 2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j} \right) \right]
\]

The Casimir Effect
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BC e.g. periodic

Casimir Effect

BC

vacuum

Φ
The Casimir Effect

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⇒ all kind of fields
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Van der Waals, Lifschitz theory

- Dynamical CE ⇐
- Lateral CE
- Extract energy from vacuum
- CE and the cosmological constant ⇐
The standard approach
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$$\rightarrow$$ Casimir force: calculated by computing change in zero point energy of the EM field
The standard approach

⇒ Casimir force: calculated by computing change in zero point energy of the em field

⇒ But Casimir effects can be calculated as $S$-matrix elements:
Feynman diagrams with ext. lines
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In modern language the Casimir energy can be expressed in terms of the trace of the Greens function for the fluctuating field in the background of interest (conducting plates)

$$\mathcal{E} = \frac{\hbar}{2\pi} \text{Im} \int d\omega \ \text{Tr} \int d^3x \left[ \mathcal{G}(x, x, \omega + i\epsilon) - \mathcal{G}_0(x, x, \omega + i\epsilon) \right]$$
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$G$ full Greens function for the fluctuating field

$G_0$ free Greens function

Trace is over spin

4th Sakharov Conference, Moscow, May 21, 2009 – p. 12/3
$E_C = \langle \text{plates} \rangle - \langle \text{no plates} \rangle$
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change in the density of states due to the background
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\[ \Rightarrow \text{“Experimental confirmation of the Casimir effect does not establish the reality of zero point fluctuations” [R. Jaffe et. al.]} \]
The Dynamical Casimir Effect

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- Moving mirrors modify structure of quantum vacuum

- Creation and annihilation of photons; once mirrors return to rest, some produced photons may still remain: flux of radiated particles
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- The dissipative part we obtain agrees with the other methods. But those have problems with the reactive part, which in general yields a non-positive energy
Hamiltonian method for neutral Klein-Gordon field in a cavity $\Omega_t$, with boundaries moving at a certain speed $v << c$, $\epsilon = v/c$

(of order $10^{-8}$ in Kim, Brownell, Onofrio, PRL 96 (2006) 200402)
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$$\mathcal{L}(t, x) = \frac{1}{2} \left[ (\partial_t \phi)^2 - |\nabla_x \phi|^2 \right], \quad \forall x \in \Omega_t \subset \mathbb{R}^n, \forall t \in \mathbb{R}$$
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- **Hamiltonian.** Transform moving boundary into fixed one by (non-conformal) change of coordinates
  \[ R: (\bar{t}, y) \rightarrow (t(\bar{t}, y), x(\bar{t}, y)) = (\bar{t}, R(\bar{t}, y)) \]
  transform $\Omega_t$ into a fixed domain $\tilde{\Omega}$
  \[ \tilde{\Omega}: (t(\bar{t}, y), x(\bar{t}, y)) = R(\bar{t}, y) = (\bar{t}, R(\bar{t}, y)) \]
  (with $\bar{t}$ the new time)
CASE OF A SINGLE, PARTIALLY TRANSMITTING MIRROR
Case of a single, partially transmitting mirror

Seminal Davis-Fulling model [PRSL A348 (1976) 393] renormalized energy negative while the mirror moves: cannot be considered as the energy of the produced particles at time $t$ [cf. paragraph after Eq. (4.5)]
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Trajectory $(t, \epsilon g(t))$. When mirror at rest, scattering described by matrix $S(\omega) = \begin{pmatrix} s(\omega) & r(\omega) e^{-2i\omega L} \\ r(\omega) e^{2i\omega L} & s(\omega) \end{pmatrix}$

$\implies$ $S$ matrix is taken to be: $(x = L$ position of the mirror)
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$\rightarrow$ Real in the temporal domain: $S(-\omega) = S^*(\omega)$
$\rightarrow$ Causal: $S(\omega)$ is analytic for $\text{Im} (\omega) > 0$
$\rightarrow$ Unitary: $S(\omega)S^\dagger(\omega) = \text{Id}$
$\rightarrow$ The identity at high frequencies: $S(\omega) \rightarrow \text{Id}$, when $|\omega| \rightarrow \infty$
s$(\omega)$ and $r(\omega)$ meromorphic (cut-off) functions
(material’s permitivity and resistivity)
RESULTS ARE REWARDING:
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In our Hamiltonian approach

\[
\langle \hat{F}_{Ha}(t) \rangle = -\frac{\epsilon}{2\pi^2} \int_0^\infty \int_0^\infty \frac{d\omega d\omega'}{\omega + \omega'} \text{Re} \left[ e^{-i(\omega+\omega')t} \hat{g}\theta_t(\omega + \omega') \right] \\
\times \left[ |r(\omega) + r^*(\omega')|^2 + |s(\omega) - s^*(\omega')|^2 \right] + \mathcal{O}(\epsilon^2)
\]

Note this integral diverges for a perfect mirror \((r \equiv -1, s \equiv 0, \text{ideal case})\), but nicely converges for our partially transmitting (physical) one where \(r(\omega) \rightarrow 0, s(\omega) \rightarrow 1, \text{as } \omega \rightarrow \infty\)
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\[ \implies \text{Two mirrors; higher dimensions; fields of any kind} \]
Quantum Vacuum Fluct’s & the CC

The main issue: S.A. Fulling et. al., hep-th/070209v2

energy *ALWAYS* gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor

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It affects cosmology: \( \tilde{T}_{\mu\nu} \) excitations above the vacuum
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Idea: zero point fluctuations can contribute to the cosmological constant Ya.B. Zeldovich ’68
**CC PROBLEM**

- Relativistic field: collection of harmonic oscill’s (scalar field)

\[ E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = k^2 + m^2/\hbar^2, \quad k = 2\pi/\lambda \]
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- Evaluating in a box and putting a cut-off at maximum \( k_{max} \) corresp’ng to QFT physics (e.g., Planck energy)

\[ \rho \sim \frac{\hbar k_{Planck}^4}{16\pi^2} \sim 10^{123} \rho_{obs} \]

kind of a modern (and thick!) aether  

R. Caldwell, S. Carroll, ...
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- What we do consider —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:

\[ \implies \text{kind of cosmological Casimir effect} \]
Cosmolog Imprint of the Casimir Eff’t?

Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}\text{mm}$ dims, bulk vs brane Susy breaking scales
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We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:
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We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:

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(b) dS & AdS worldbranes

(c) supergraviton theories (discret dims, deconstr)
The Braneworld Case

1. Braneworld may help to solve:
   - the hierarchy problem
   - the cosmological constant problem

2. Presumably, the bulk Casimir effect will play a role in the construction (radion stabilization) of braneworlds
   - Bulk Casimir effect (effective potential) for a conformal or massive scalar field
   - Bulk is a 5-dim AdS or dS space with 2/1 4-dim dS brane (our universe)
   - Consistent with observational data even for relatively large extra dimension

Previous work:
   - → flat space brane
   - → bulk conformal scalar field
   - → conclusion: no CE

We used zeta regularization at full power, with positive results!

EE, S Nojiri, SD Odintsov, S Ogushi, Phys Rev D67 (2003) 063515 *Casimir effect in de Sitter and Anti-de Sitter braneworlds*  
EE, SD Odintsov, AA Saharian 0902.0717 *Repulsive Casimir effect from extra dimensions and Robin BC: from branes to pistons*
Casimir eff in brworl’s w large extra dim

Casimir energy for massive scalar field with an arbitrary curvature coupling, obeying Robin BCs on two codim-1 parallel plates embedded in background spacetime \( R^{(D_1-1,1)} \times \Sigma \), \( \Sigma \) compact internal space
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Most general case: constants in the BCs different for the two plates
It is shown that Robin BCs with different coefficients are necessary to obtain repulsive Casimir forces
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Robin type BCs are an extension of Dirichlet and Neumann’s most suitable to describe physically realistic situations.
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Robin type BCs are an **extension** of Dirichlet and Neumann’s → most suitable to describe physically **realistic** situations.

Genuinely appear in: vacuum effects for a **confined charged scalar** field in external fields [Ambjørn ea 83], spinor and gauge field theories, quantum gravity and supergravity [Luckock ea 91] Can be made **conformally invariant**, while purely-Neumann conditions cannot → needed for **conformally invariant** theories with boundaries, to preserve this invariance.
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Quantum scalar field with Robin BCs on boundary of cavity violates Bekenstein’s entropy-to-energy bound near certain points in the space of the parameter defining the boundary condition [Solodukhin 01]
Robin BCs can model the \textit{finite penetration} of the field through the boundary: the ‘\textit{skin-depth}’ param related to Robin coefficient \cite{Mostep ea 85,Lebedev 01}

Casimir forces between the \textit{boundary planes} of films \cite{Schmidt ea 08}
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For arbitrary internal space, \textit{interaction part of the Casimir energy} given by

$$\Delta E_{[a_1, a_2]} = \frac{(4\pi)^{-D_1/2}}{\Gamma(D_1/2)} \sum_{\beta} \int_{m_\beta}^\infty dx \frac{x^2 - m_\beta^2}{D_1/2-1} \times \ln \left[ 1 - \frac{(\beta_1 x + 1)(\beta_2 x + 1)}{(\beta_1 x - 1)(\beta_2 x - 1)} e^{-2ax} \right]$$

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$$\times \ln \left[ 1 - \frac{(\beta_1 x + 1)(\beta_2 x + 1)}{(\beta_1 x - 1)(\beta_2 x - 1)} e^{-2ax} \right] (*)$$

For Dirichlet and Neumann BCs on both plates this leads to

$$\Delta E^{(J,J)}_{[a_1, a_2]} = -\frac{2a^{-D_1}}{(8\pi)(D_1+1)/2} \sum_{\beta} \sum_{n=1}^{\infty} \frac{f(D_1+1)/2(2n a m_{\beta})}{n^{D_1+1}}$$

with $f_\nu(z) = z^\nu K_\nu(z)$ → energy always negative
For Dirichlet BC on one plate and Neumann on the other, the interaction component of the vacuum energy is

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\Delta E_{[a_1,a_2]}^{(D,N)} = \frac{(4\pi)^{-D_1/2}a}{\Gamma(D_1/2 + 1)} \sum_\beta \int_{m_\beta}^\infty dx \frac{(x^2 - m_\beta^2)^{D_1/2}}{e^{2ax} + 1}
\]

\[
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positive for all values of the inter-plate distance
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In the case of a conformally coupled massless field on the background of a spacetime conformally related to the one described by the line element

\[
ds^2 = g_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu - \gamma_{il} dx^i dx^l
\]

\(\eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1)\) metric of \((D_1 + 1)\)-dim Minkowski st and \(X^i\) coordinates of \(\Sigma\), with the conformal factor \(\Omega^2(x^{D_1})\). Interaction part of Casimir energy is given (*), with coeffs \(\beta_j\) related to coeffs of the Robin BCs

\[
(1 + \beta_j n^M \nabla_M) \varphi(x) = [1 + (-1)^j \Omega_j^{-1} \beta_j \partial_{D_1}] \varphi(x) = 0, \quad \Omega_j = \Omega(x^{D_1}_j)
\]

& conformal factor \(\beta_j = \left[\Omega_j + (-1)^j \frac{D-1}{2\Omega_j} \beta_j \Omega_j'\right]^{-1} \beta_j, \quad \Omega_j' = \Omega_j'(x^{D_1}_j)\)
In Randall-Sundrum 2-brane model with compact internal space, the Robin coefficients are \( \beta_j^{-1} = (-1)^j \frac{c_j}{2} - 2D\zeta / r_D \), \( c_1, c_2 \) mass parameters in the surface action of the scalar field for the left and right branes, respectively. The vacuum energy can have a minimum, for the stable equilibrium point. Can be used in braneworld models for the stabilization of the radion field.
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We have considered a piston-like geometry, introducing a third plate (then this plate is sent to infinity) Casimir force:

\[
P = -\frac{2(4\pi)^{-D_{1}/2}}{V_{\Sigma} \Gamma(D_{1}/2) a^{D_{1}+1}} \sum_{\beta} \int_{a m_{\beta}}^{\infty} dx \frac{x^{2}(x^{2} - a^{2}m_{\beta}^{2})^{D_{1}/2-1}}{(b_{1}x-1)(b_{2}x-1)(b_{1}x+1)(b_{2}x+1)e^{2x} - 1}
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With independence of the geometry of the internal space, the force is attractive for Dirichlet or Neumann boundary conditions on both plates.

\[
P^{(J,J)} = -\frac{2(4\pi)^{-D_1/2}}{V_\Sigma \Gamma(D_1/2)} \sum_{\beta} \int_{m_\beta}^{\infty} dx \frac{x^2 (x^2 - m_\beta^2)^{D_1/2-1}}{e^{2ax} - 1} = \frac{2a^{-D_1-1}}{(8\pi)^{(D_1+1)/2} V_\Sigma} \sum_{\beta} \sum_{n=1}^{\infty} \frac{1}{n^{D_1+1}} \left[ f(D_1+1)/2 (2na m_\beta) - f(D_1+3)/2 (2na m_\beta) \right] \]

\( J = D, N \), and repulsive for Dirichlet BC on one plate and Neumann on the other, a monotonic function of the distance.
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In absence of zero modes (case of twisted boundary conditions along compactified dimensions), Casimir forces are exponentially suppressed in the limit of small size of the internal space. For small values of the inter-plate distance the Casimir forces are attractive, independently of the values of the Robin coefficients, except for the case of Dirichlet boundary conditions on one plate and non-Dirichlet boundary conditions on the other. In this latter case, the Casimir force is repulsive at small distances.
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Interesting remark: this property could be used in the proposal of a Casimir experiment with the purpose to carry out an explicit detailed observation of ‘large’ extra dimensions as allowed by some models of particle physics.
Gravity Eqs as Eqs of State: f(R) Case

The cosmological constant as an "integration constant"

T. Padmanabhan; D. Blas, J. Garriga, E. Alvarez ...

Unimodular Gravity
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- C. Eling, R. Guedens, T. Jacobson [PRL2006]: extension to polynomial $f(R)$ gravity but as non-equilibrium thermodyn. Also Erik Verlinde (private discussions)
Jacobson’s argument: basic thermodynamic relation

\[ \delta Q = T \delta S \]

- entropy proportional to variation of the horizon area: \( \delta S = \eta \delta A \)
- local temperature \( T \) defined as Unruh temp: \( T = \hbar k / 2\pi \)
- functional dependence of \( S \) wrt energy and size of system
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Key point in our generalization: the definition of the local entropy (Iyer+Wald 93: local boost inv, Noether charge)

\[ S = -2\pi \int_\Sigma E_{R}^{pqrs} \epsilon_{pq} \epsilon_{rs}, \quad \delta S = \delta (\eta_e A) \]

\( \eta_e \) is a function of the metric and its deriv’s to a given order

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Case of \( f(R) \) gravities: \( L = f(R, \nabla^n R) \)
Also the concept of an effective Newton constant for graviton exchange (effective propagator)

\[
\frac{1}{8\pi G_{\text{eff}}} = E^{pqrs}_R \epsilon_{pq} \epsilon_{rs} = \frac{\partial f}{\partial R} (g^{pr} g^{qs} - g^{qr} g^{ps}) \epsilon_{pq} \epsilon_{rs}
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Final result, for \(f(R)\) gravities:

*the local field equations can be thought of as an equation of state of equilibrium thermodynamics* (as in the GR case)
Jacobson’s argument non-trivially extended to $f(R)$ gravity field equations as EoS of local space-time thermodynamics
EE, P. Silva, Phys Rev D78, 061501(R) (2008), arXiv:0804.3721v2
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RM Wald PRD1993; V Iyer, RM Wald PRD1994
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S-F Wu, G-H Yang, P-M Zhang, arXiv:0805.4044, direct extension of our results to Brans-Dicke and scalar-tensor gravities
Perspectives:

Shear viscosity to entropy density ratio: $\frac{\eta}{s} \geq 1/4\pi$
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- Hydrodynamic correspondence. Heavy ion experiments: correspond Navier-Stokes coefficients determined from measurement. And for BH physics.
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   (i) # particles very large; (ii) # excit of a particle v.l.
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- Good candidates to violate bound: strong continuum spectrum [A Jakovac, D Nogradi 0810.4181] lattice calc 0.056 vs 0.0796
Perspectives:

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AdS/CFT corresp $\rightarrow$ N=4 SUSY Y-M [DT Son, AO Starinets, ...]
QCD non-conf, but at very high $T$ approx conf (deconfining phase)

Bound saturated for N=4 SUSY Y-M at large 't Hooft coupl & large N lim, and first corrections are positive!

Hydrodyn corresp. Heavy ion experiments: corresp Navier-Stokes coeefs determined from measurement. And for BH physics.

Lower bound $\sim E-t$ uncert relat (Heis), using quasi-particle picture

Counterexamp to bound: (in gener grav case? R Brustein, C Eling) (i) # particles very large; (ii) # excit of a particle v.l.

Good candidates to violate bound: strong continuum spectrum [A Jakovac, D Nogradi 0810.4181] lattice calc 0.056 vs 0.0796

Thanks for your attention