Repulsive Casimir Forces from Additional Dimensions

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Outline

On Einstein’s Cosmological Constant
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- Casimir Effect & the $\zeta$ Function Method
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- CE and Accelerated Expansion (Dark Energy)
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- Gravity Equations as Equations of State
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With THANKS to:
S. Carloni, G. Cognola, J. Haro, S.D. Odintsov, A. Saharian, P.J. Silva, S. Zerbini, ...
On Einstein’s Cosmological Constant

Our universe seems to be spatially flat and to possess a non-vanishing cosmological constant

- For cosmologists and general relativists: *a great mistake* (Einstein)

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \left( \frac{8\pi G}{c^4} \right) T_{\mu\nu} + \lambda g_{\mu\nu} \]

- For elementary particle physicists: *a great embarrassment* no way to get rid off (Coleman, Weinberg, Polchinski)

- The cc $\Lambda$ is indeed a peculiar quantity

  - has to do with cosmology  Einstein’s eqs., FRW universe
  - has to do with the local structure of elementary particle physics stress-energy density $\mu$ of the vacuum

\[ L_{cc} = \int d^4 x \sqrt{-g} \mu^4 = \frac{1}{8\pi G} \int d^4 x \sqrt{-g} \lambda \]

In other words: two contributions on the same footing (Zel’dovich, 68)

\[ \frac{\Lambda c^2}{8\pi G} + \frac{1}{\text{Vol}} \frac{\hbar c}{2} \sum_i \omega_i \]
Einstein Equations (1915-17): \[ G_{\mu\nu} - \lambda g_{\mu\nu} = -(8\pi G/c^4)T_{\mu\nu} \]

Geometry = Energy-Matter

\( G_{\mu\nu} \) linear combination of the metric \( g_{\mu\nu} \) and 1st & 2nd derivatives

\( T_{\mu\nu} \) energy-momentum tensor

Schwarzschild solution (1916)

\[
\begin{align*}
    ds^2 &= \left(1 - \frac{2MG}{r}\right)dt^2 - \left(1 - \frac{2MG}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2
\end{align*}
\]

Friedmann-Lemaître-Robertson-Walker (1935-36) sol (A. Friedmann 1922)

\[
\begin{align*}
    ds^2 &= dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right)
\end{align*}
\]

gen fam: homog + isotrop, \( k \) par \( \pm 1, 0 \)

Hubble ea 1923-29, Keeler Slipher Campbell 1918

One field eq looks like Newtonian eq for the gravit pot: \( \nabla^2 \phi = 4\pi G (\rho + 3p/c^2) \)

density & pressure contribute to the gravit pot \( \lambda = 8\pi G\rho_{vac}, \ p_{vac} = -\rho_{vac} c^2 \)

From the FRW metric and Einstein Eqs, an “equation of motion” of the universe

\[
H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\lambda}{3} - \frac{k}{a^2}
\]
From GR to Cosmology

With the definitions:

$$\Omega_m \equiv \frac{8\pi G}{3H^2}\rho_m, \quad \Omega_\lambda \equiv \frac{\lambda}{3H^2}, \quad \Omega_k \equiv -\frac{k}{H^2}$$

The equation of motion becomes

$$\left(\frac{da}{dt}\right)^2 = H_0^2 \left[ \Omega_m^{(0)} \frac{a}{a} + a^2 \Omega_\lambda^{(0)} + \Omega_k^{(0)} \right]$$

(the superscript (o) represents quantities measured at the present time)

In other terms, Friedmann equation in Cosmology:

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[ \Omega_R \left(\frac{a_0}{a}\right)^4 + \Omega_{NR} \left(\frac{a_0}{a}\right)^3 + \Omega_k \left(\frac{a_0}{a}\right)^2 + \Omega_\lambda \right]$$

- \(\Omega_R\) relativistic matter (\(p_R = \frac{1}{3}\rho_R; \ \rho_R \propto a^{-4}\))
- \(\Omega_{NR}\) nonrelativistic matter (\(p_{NR} = 0; \ \rho_{NR} \propto a^{-3}\))
- \(\Omega_\lambda\) cosmological constant (\(p_\lambda = -\rho_\lambda; \ \rho_\lambda = \text{const}\))

\(\Omega = \Omega_R + \Omega_{NR} + \Omega_\lambda\) total energy density (cosmic triangle)
Zero point energy

QFT vacuum to vacuum transition: $\langle 0 | H | 0 \rangle$
Zero point energy

QFT vacuum to vacuum transition: \( \langle 0 | H | 0 \rangle \)

Spectrum, normal ordering (harm oscill):

\[
H = \left( n + \frac{1}{2} \right) \lambda_n a_n a_n^\dagger
\]
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\[ \langle 0 | H | 0 \rangle = \frac{\hbar c}{2} \sum_n \lambda_n = \frac{1}{2} \text{tr} \ H = \frac{1}{2} \zeta_H^\mu (-1) \]
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Regularization + Renormalization (cut-off, dim, \( \zeta \))
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Regularization + Renormalization (cut-off, dim, \( \zeta \))

Even then: Has the final value real sense?
Existence of $\zeta_A$ for $A$ a $\Psi DO$

1. $A$ a positive-definite elliptic $\Psi DO$ of positive order $m \in \mathbb{R}^+$
2. $A$ acts on the space of smooth sections of
3. $E$, $n$-dim vector bundle over
4. $M$ closed $n$-dim manifold
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(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr} \ A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re} \ s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of $A$, $s_0 = \dim M / \text{ord} A$ abscissa of converg of $\zeta_A(s)$
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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$,

$\text{Spec } A \cap L_\theta = \emptyset$

(the Agmon-Nirenberg condition)
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(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$

(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \quad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$
Definition of Determinant

\[ H \] \[ \Psi \] DO operator \{ \varphi_i, \lambda_i \} spectral decomposition
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\[ \prod_{i \in I} \lambda_i \]

\[ \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]
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Riemann zeta func: \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \ Re \ s > 1 \) (\& analytic cont)

Definition: zeta function of \( H \)

\[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s} \]

As Mellin transform: \( \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{tr} \ e^{-tH}, \ Re s > s_0 \)

Derivative: \( \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \)
Definition of Determinant

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Weierstrass def: subtract leading behavior of \( \lambda_i \) in \( i \), as \( i \to \infty \), until series \( \sum_{i \in I} \ln \lambda_i \) converges \[ \implies \text{non-local counterterms} \]
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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...
Properties

The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut.
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- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex $s$-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C} \setminus \mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik].
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- Asymptotic expansion for the heat kernel:
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Asymptotic expansion for the heat kernel:

\[
\text{tr} e^{-tA} = \sum'_{\lambda \in \text{Spec } A} e^{-t\lambda} \\
\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0
\]

\[
\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \text{Res}_{s=s_j} \zeta_A(s), \quad s_j \notin -\mathbb{N}
\]

\[
\alpha_j(A) = \frac{(-1)^k}{k!} \left[ \text{PP} \zeta_A(-k) + \psi(k + 1) \text{Res}_{s=-k} \zeta_A(s) \right],
\]

\[
\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N} \setminus \{0\}
\]

\[
\text{PP } \phi := \lim_{s \to p} \left[ \phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]
\]
The Chowla-Selberg Expansion Formula: Basics

- **Jacobi’s identity** for the $\theta$-function

$$\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \quad q := e^{i\pi \tau}, \quad \tau \in \mathbb{C}$$

$$\theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi \tau} \theta_3 \left( \frac{z}{\tau} \mid -\frac{1}{\tau} \right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi nz), \quad z, t \in \mathbb{C}, \quad \text{Re} t > 0$$
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- **Higher dimensions:** Poisson summ formula (Riemann)

  $$\sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \tilde{f}(\vec{m})$$

  $\tilde{f}$ Fourier transform

  [Gelbart + Miller, BAMS ’03, Iwaniec, Morgan, ICM ’06]
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$$\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2n z), \quad q := e^{i\pi \tau}, \tau \in \mathbb{C}$$

$$\theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi \tau} \theta_3\left(\frac{z}{\tau}\left|\frac{-1}{\tau}\right)\right.$$ equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2/t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\pi^2 n^2/t} \cos(2\pi n z), \quad z, t \in \mathbb{C}, \text{ Re} t > 0$$

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\[\tilde{f}\] Fourier transform

[Gelbart + Miller, BAMS ’03, Iwaniec, Morgan, ICM ’06]

- Truncated sums $\rightarrow$ asymptotic series
Extended CS Formulas (ECS)

Consider the zeta function \((\text{Re} s > p/2, A > 0, \text{Re} q > 0)\)

\[
\zeta_{A, \vec{c}, q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} \prime \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{n} \in \mathbb{Z}^p} \prime \left[ Q (\vec{n} + \vec{c}) + q \right]^{-s}
\]

prime: point \(\vec{n} = \vec{0}\) to be excluded from the sum

(inescapable condition when \(c_1 = \cdots = c_p = q = 0\))

\[
Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}
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**Case** \(q \neq 0 (\text{Re} q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\text{det} A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2} \pi^s q^{-s/2+p/4}}{\sqrt{\text{det} A} \Gamma(s)}
\]

\[
\times \sum_{\vec{m} \in \mathbb{Z}_{1/2}^p}^{'} \cos(2\pi \vec{m} \cdot \vec{c}) (\vec{m}^T A^{-1} \vec{m})^{s/2-p/4} K_{p/2-s} \left(2\pi \sqrt{2q} \vec{m}^T A^{-1} \vec{m}\right)
\]

[1] ECS1
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\]

\[
\times \sum_{\vec{m} \in \mathbb{Z}_{1/2}^p} \cos(2\pi \vec{m} \cdot \vec{c}) \left( \vec{m}^T A^{-1} \vec{m} \right)^{s/2-p/4} K_{p/2-s} \left( 2\pi \sqrt{2q \vec{m}^T A^{-1} \vec{m}} \right)
\]

**Pole:** \(s = p/2\)

**Residue:**

\[
\text{Res}_{s=p/2} \zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}
\]
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.
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• Exhibits singularities (simple poles) of the meromorphic continuation — with the corresponding residua — explicitly
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Only condition on matrix $A$: corresponds to (non negative) quadratic form, $Q$. Vector $\vec{c}$ arbitrary, while $q$ is (to start) a non-neg constant
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$K_\nu$ modified Bessel function of the second kind and the subindex $1/2$ in $\mathbb{Z}^{p}_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose first non-zero component is positive].
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Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[ \pi^{j/2} a_j^{j/2-s} \Gamma \left( s - \frac{j}{2} \right) \zeta_R(2s-j) + ight.$$ 

$$4\pi^s a_j^{-\frac{j}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j} n^{j/2-s} \left( \vec{m}_j^t A_j^{-1} \vec{m}_j \right)^{s/2-j/4} K_{j/2-s} \left( 2\pi n \sqrt{a_p-j \vec{m}_j^t A_j^{-1} \vec{m}_j} \right)$$

[EC3d]
The Casimir Effect
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BC e.g. periodic

Casimir Effect

BC

vacuum

Φ
The Casimir Effect

BC  e.g. periodic
⇒ all kind of fields

vacuum

Casimir Effect
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Universal process:
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Universal process:
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Van der Waals, Lifschitz theory

- Dynamical CE
- Lateral CE
- Extract energy from vacuum
- CE and the cosmological constant
The Dynamical Casimir Effect

The Dynamical Casimir Effect


- Moving mirrors modify structure of quantum vacuum
- Creation and annihilation of photons; once mirrors return to rest, some produced photons may still remain: flux of radiated particles
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Moore; Razavy, Terning; Johnston, Sarkar; Dodonov et al; Plunien et al; Barton, Eberlein, Calogeracos; Ford, Vilenkin; Jaeckel, Reynaud, Lambrechts; Brevik, Milton et al; Dalvit, Maia-Neto et al; Law; Parentani, ...
A CONSISTENT APPROACH:

J. Haro & E.E., PRL 97 (2006); arXiv:0705.0597
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- Such force is split into two parts: a dissipative force whose work equals minus the energy of the particles that remain & a reactive force vanishing when the mirrors return to rest
- The dissipative part we obtain agrees with the other methods. But those have problems with the reactive part, which in general yields a non-positive energy

⇒ EXPERIMENT
SOME DETAILS OF THE METHOD

Hamiltonian method for neutral Klein-Gordon field in a cavity $\Omega_t$, with boundaries moving at a certain speed $v << c$, $\epsilon = v/c$

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  $$\mathcal{L}(t, x) = \frac{1}{2} \left[ (\partial_t \phi)^2 - |\nabla_x \phi|^2 \right], \quad \forall x \in \Omega_t \subset \mathbb{R}^n, \quad \forall t \in \mathbb{R}$$
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- Hamiltonian. Transform moving boundary into fixed one by (non-conformal) change of coordinates

$$\mathcal{R} : (\bar{t}, y) \rightarrow (t(\bar{t}, y), x(\bar{t}, y)) = (\bar{t}, \mathbf{R}(\bar{t}, y))$$

transform $\Omega_t$ into a fixed domain $\tilde{\Omega}$

$$\widetilde{\Omega} : (t(\bar{t}, y), x(\bar{t}, y)) = \mathcal{R}(\bar{t}, y) = (\bar{t}, \mathbf{R}(\bar{t}, y))$$

(with $\bar{t}$ the new time)
CASE OF A SINGLE, PARTIALLY TRANSMITTING MIRROR
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Seminal Davis-Fulling model [PRSL A348 (1976) 393] renormalized energy negative while the mirror moves: cannot be considered as the energy of the produced particles at time $t$ [cf. paragraph after Eq. (4.5)]
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Our interpretation: a perfectly reflecting mirror is non-physical. Consider, instead, a partially transmitting mirror, transparent to high frequencies (math. implementation of a physical plate).
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Trajectory $(t, \epsilon g(t))$. When mirror at rest, scattering described by matrix $S(\omega) = \begin{pmatrix} s(\omega) & r(\omega) e^{-2i\omega L} \\ r(\omega) e^{2i\omega L} & s(\omega) \end{pmatrix}$

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\]
\(\Rightarrow S\) matrix is taken to be: \((x = L \text{ position of the mirror})\)
\(\rightarrow\) Real in the temporal domain: \(S(-\omega) = S^*(\omega)\)
\(\rightarrow\) Causal: \(S(\omega)\) is analytic for \(\text{Im} (\omega) > 0\) [S Reynaud]
\(\rightarrow\) Unitary: \(S(\omega)S^\dagger(\omega) = \text{Id}\)
\(\rightarrow\) The identity at high frequencies: \(S(\omega) \to \text{Id}, \text{when} |\omega| \to \infty\)
\(s(\omega)\) and \(r(\omega)\) meromorphic (cut-off) functions
(material’s permittivity and resistivity)
RESULTS ARE REWARDING:
In our Hamiltonian approach

\[ \langle \hat{F}_{Ha}(t) \rangle = -\frac{\epsilon}{2\pi^2} \int_0^\infty \int_0^\infty \frac{d\omega d\omega' \omega \omega'}{\omega + \omega'} \text{Re} \left[ e^{-i(\omega + \omega')t} \hat{\theta}_t(\omega + \omega') \right] \]

\[ \times \left[ |r(\omega) + r^*(\omega')|^2 + |s(\omega) - s^*(\omega')|^2 \right] + \mathcal{O}(\epsilon^2) \]

Note this integral diverges for a perfect mirror \((r \equiv -1, s \equiv 0, \text{ideal case})\), but nicely converges for our partially transmitting (physical) one where \(r(\omega) \to 0, s(\omega) \to 1\), as \(\omega \to \infty\).
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\[\rightarrow\] Two mirrors; higher dimensions; fields of any kind
The main issue: energy ALWAYS gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor \[ \langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu} \]
Quantum Vacuum Fluct’s & the CC

The main issue: S.A. Fulling et. al., hep-th/070209v2

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Appears on the rhs of Einstein’s equations: [E Mottola]

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It affects cosmology: \( \tilde{T}_{\mu\nu} \) excitations above the vacuum
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\[ \Lambda = \left(2.14 \pm 0.13 \times 10^{-3} \text{ eV}\right)^4 \sim 4.32 \times 10^{-9} \text{ erg/cm}^3 \]
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Idea: zero point fluctuations can contribute to the cosmological constant Ya.B. Zeldovich ’68
CC PROBLEM

Relativistic field: collection of harmonic oscill’s (scalar field)

\[ E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \quad \omega = k^2 + \frac{m^2}{\hbar^2}, \quad k = \frac{2\pi}{\lambda} \]
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  \[ \rho \sim \frac{\hbar k_{\text{Planck}}}{16\pi^2} \sim 10^{123} \rho_{\text{obs}} \]

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- What we do consider —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:

\[ \implies \text{kind of cosmological Casimir effect} \]
Cosmolog Imprint of the Casimir Eff’t?

Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}\,\text{mm}$ dims, bulk vs brane Susy breaking scales
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(a) small and large compactified scales
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We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:

(a) small and large compactified scales
(b) dS & AdS worldbranes
Cosmolog Imprint of the Casimir Eff’t?

Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

- L. Parker & A. Raval, VCDM, vacuum energy density
- C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}\text{mm}$ dims, bulk vs brane Susy breaking scales
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We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:

- (a) small and large compactified scales
- (b) dS & AdS worldbranes
- (c) supergraviton theories (discret dims, deconstr)
The Braneworld Case

1. Braneworld may help to solve:
   - the hierarchy problem
   - the cosmological constant problem

2. Presumably, the bulk Casimir effect will play a role in the construction (radion stabilization) of braneworlds
   - Bulk Casimir effect (effective potential) for a conformal or massive scalar field
   - Bulk is a 5-dim AdS or dS space with 2/1 4-dim dS brane (our universe)
   - Consistent with observational data even for relatively large extra dimension

Previous work:
- flat space brane
- bulk conformal scalar field
- conclusion: no CE

We used zeta regularization at full power, with positive results!

EE, S Nojiri, SD Odintsov, S Ogushi, Phys Rev D67 (2003) 063515 Casimir effect in de Sitter and Anti-de Sitter braneworlds
EE, SD Odintsov, AA Saharian 0902.0717 Repulsive Casimir effect from extra dimensions and Robin BC: from branes to pistons
The Sign of the Casimir Force

Many papers dealing on this issue: here just short account
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More general results: Kenneth, Klich, PRL 97, 160401 (2006) a mirror pair of dielectric bodies always attract each other
E.g. \( \exists \) correlation inequality: \( \langle f \Theta(f) \rangle > 0 \)

\( \Theta \) reflection with respect to a 3-dim hyperplane in \( \mathbb{R}^4 \)

the action of \( \Theta \) on \( f \) is anti-unitary \( \Theta(cf) = c^*\Theta(f) \)
E.g. ⋄ correlation inequality: $\langle f \Theta(f) \rangle > 0$

$\Theta$ reflection with respect to a 3-dim hyperplane in $\mathbb{R}^4$

the action of $\Theta$ on $f$ is anti-unitary $\Theta(cf) = c^* \Theta(f)$

The existence of the reflection operator $\Theta$ is a consequence of unitarity only, and makes no assumptions about the discrete $C, P, T$ symmetries
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\begin{itemize}
\item \(\Theta\) reflection with respect to a 3-dim hyperplane in \(\mathbb{R}^4\)
\item the action of \(\Theta\) on \(f\) is anti-unitary \(\Theta(cf) = c^*\Theta(f)\)
\end{itemize}

The existence of the reflection operator \(\Theta\) is a consequence of \textit{unitarity} only, and makes no assumptions about the discrete \(C, P, T\) symmetries

Boyer’s result does not contradict the theorem, since cutting an elastic shell into two rigid hemispheres is a \textit{mathematically singular} operation (which introduces divergent edge contributions)
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Θ reflection with respect to a 3-dim hyperplane in $R^4$

the action of $Θ$ on $f$ is anti-unitary $Θ(cf) = c^*Θ(f)$

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- mirror probes in a Fermi sea (chemical-potential term), eg when electron-gas fluctuations become important
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Theorem does not apply for

- mirror probes in a Fermi sea (chemical-potential term), eg when electron-gas fluctuations become important
- periodic BCs for fermions
- Robin BCs in general
Casimir eff in brworl’s w large extra dim

Casimir energy for massive scalar field with an arbitrary curvature coupling, obeying Robin BCs on two codim-1 parallel plates embedded in background spacetime $R^{(D_1-1,1)} \times \Sigma$, $\Sigma$ compact internal space
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Most general case: constants in the BCs different for the two plates. It is shown that Robin BCs with different coefficients are necessary to obtain repulsive Casimir forces.
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Robin type BCs are an extension of Dirichlet and Neumann’s
$\implies$ most suitable to describe physically realistic situations
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Genuinely appear in: $\rightarrow$ vacuum effects for a confined charged scalar field in external fields [Ambjørn ea 83],
$\rightarrow$ spinor and gauge field theories,
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Quantum scalar field with Robin BCs on boundary of cavity violates Bekenstein’s entropy-to-energy bound near certain points in the space of the parameter defining the boundary condition [Solodukhin 01]
Robin BCs can model the finite penetration of the field through the boundary: the ‘skin-depth’ param related to Robin coefficient [Mostep ea 85, Lebedev 01] Casimir forces between the boundary planes of films [Schmidt ea 08]
Robin BCs can model the **finite penetration** of the field through the boundary: the ‘skin-depth’ param related to Robin coefficient [Mostep ea 85, Lebedev 01] Casimir forces between the **boundary planes** of films [Schmidt ea 08]

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For arbitrary internal space, interaction part of the Casimir energy given by

\[ \Delta E_{[a_1,a_2]} = \frac{(4\pi)^{-D_1/2}}{\Gamma(D_1/2)} \sum_{\beta} \int_{m_{\beta}}^{\infty} dx \, x(x^2 - m_{\beta}^2)^{D_1/2-1} \]

\[ \times \ln \left[ 1 - \frac{(\beta_1 x + 1)(\beta_2 x + 1)}{(\beta_1 x - 1)(\beta_2 x - 1)} e^{-2ax} \right] \quad (*) \]
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$$\times \ln \left[1 - \frac{(\beta_1 x + 1)(\beta_2 x + 1)}{(\beta_1 x - 1)(\beta_2 x - 1)} e^{-2ax}\right]$$

For Dirichlet and Neumann BCs on both plates this leads to

$$\Delta E_{[a_1,a_2]}^{(J,J)} = -\frac{2a^{-D_1}}{(8\pi)(D_1+1)/2} \sum_\beta \sum_{n=1}^{\infty} \frac{f(D_1+1)/2(2n\text{am}_\beta)}{n^{D_1+1}}$$

with $$f_\nu(z) = z^\nu K_\nu(z)$$ → energy always negative
For Dirichlet BC on one plate and Neumann on the other, the interaction component of the vacuum energy is

\[
\Delta E^{(D,N)}_{[a_1,a_2]} = \frac{(4\pi)^{-D_1/2}a}{\Gamma(D_1/2 + 1)} \sum_\beta \int_{m_\beta}^\infty dx \frac{(x^2 - m_\beta^2)^{D_1/2}}{e^{2ax} + 1}
\]

\[
= - \frac{2a^{-D_1}}{(8\pi)^{(D_1+1)/2}} \sum_\beta \sum_{n=1}^{\infty} \frac{f(D_1+1)/2(2nam_\beta)}{(-1)^n n n^{D_1+1}}
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positive for all values of the inter-plate distance
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positive for all values of the inter-plate distance.

In the case of a conformally coupled massless field on the background of a spacetime conformally related to the one described by the line element

\[
ds^2 = g_{MN} dx^M dx^N = \eta_{\mu \nu} dx^\mu dx^\nu - \gamma_{ij} dX^i dX^j
\]

\(\eta_{\mu \nu} = \text{diag}(1,-1,\ldots,-1)\) metric of \((D_1 + 1)\)-dim Minkowski st and \(X^i\) coordinates of \(\Sigma\), with the conformal factor \(\Omega^2(x^{D_1})\). Interaction part of Casimir energy is given (*), with coeffs \(\beta_j\) related to coeffs of the Robin BCs

\[
(1 + \beta_j n^M \nabla_M) \varphi(x) = [1 + (-1)^j \Omega_j^{-1} \beta_j \partial_{D_1}] \varphi(x) = 0, \Omega_j = \Omega(x_j^{D_1})
\]

& conformal factor \(\beta_j = \left[ \Omega_j + (-1)^j \frac{D_1-1}{2\Omega_j} \beta_j \Omega_j' \right]^{-1} \beta_j, \Omega_j' = \Omega_j'(x_j^{D_1})\)
In Randall-Sundrum 2-brane model with compact internal space, the Robin coefficients are $\bar{\beta}_j^{-1} = (-1)^j c_j / 2 - 2D\zeta / r_D$, $c_1, c_2$ mass parameters in the surface action of the scalar field for the left and right branes, respectively. The vacuum energy can have a minimum, for the stable equilibrium point. Can be used in braneworld models for the stabilization of the radion field.
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We have considered a piston-like geometry, introducing a third plate (then this plate is sent to infinity)

\[
P = -\frac{2(4\pi)^{-D_1/2}}{V\Sigma\Gamma(D_1/2)a^{D_1+1}} \sum \int_{am_\beta}^\infty \frac{dx}{(b_1 x-1)(b_2 x-1)} \frac{x^2(x^2 - a^2 m^2) D_1/2-1}{(b_1 x+1)(b_2 x+1)e^{2x} - 1}
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We have considered a piston-like geometry, introducing a third plate (then this plate is sent to infinity). Casimir force

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\]

With independence of the geometry of the internal space, the force is attractive for Dirichlet or Neumann boundary conditions on both plates

\[
P^{(J,J)} = -\frac{2(4\pi)^{-D_1/2}}{V_\Sigma \Gamma(D_1/2)} \sum_\beta \int_0^\infty \, dx \frac{x^2 (x^2 - m^2_\beta)^{D_1/2-1}}{e^{2ax} - 1}
= \frac{2a^{-D_1-1}}{(8\pi)^{(D_1+1)/2} V_\Sigma} \sum_\beta \sum_{n=1}^\infty \frac{1}{n^{D_1+1}} \left[ f(D_1+1)/2(2nam_\beta) - f(D_1+3)/2(2nam_\beta) \right]
\]

\( J = D, N \), and repulsive for Dirichlet BC on one plate and Neumann on the other, a monotonic function of the distance.
For general Robin BCs the Casimir force can be either attractive (negative $P$) or repulsive (positive $P$), depending on the Robin coefficients and distance between plates.
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For small values of the size of internal space, in models with zero modes along the internal space, main contribution to Casimir force comes from the zero modes: contributions of non-zero modes are exponentially suppressed.
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In this limit, to leading order we recover the standard result for the Casimir force between two plates in $(D_1 + 1)$-dim Minkowski spacetime.
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In absence of zero modes (case of twisted boundary conditions along compactified dimensions), Casimir forces are exponentially suppressed in the limit of small size of the internal space. For small values of the inter-plate distance the Casimir forces are attractive, independently of the values of the Robin coefficients, except for the case of Dirichlet boundary conditions on one plate and non-Dirichlet boundary conditions on the other.

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In this latter case, the Casimir force is repulsive at small distances.

Interesting remark: this property could be used in the proposal of a Casimir experiment with the purpose to carry out an explicit detailed observation of ‘large’ extra dimensions as allowed by some models of particle physics.
Gravity Eqs as Eqs of State: f(R) Case

The cosmological constant as an “integration constant”

T. Padmanabhan; D. Blas, J. Garriga, E. Alvarez ...

Unimodular Gravity Also I Shapiro, J Solà,... cc RG flow
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- C. Eling, R. Guedens, T. Jacobson [PRL2006]: extension to polynomial $f(R)$ gravity but as non-equilibrium thermodyn.
  Also Erik Verlinde (private discussions)
Jacobson’s argument: basic thermodynamic relation

\[ \delta Q = T \delta S \]

- entropy proport to variation of the horizon area: \( \delta S = \eta \delta A \)
- local temperature \( T \) defined as Unruh temp: \( T = \frac{\hbar k}{2\pi} \)
- functional dependence of \( S \) wrt energy and size of system
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- functional dependence of \( S \) wrt energy and size of system

Key point in our generalization: the definition of the local entropy (Iyer+Wald 93: local boost inv, Noether charge)

\[ S = -2\pi \int_{\Sigma} E_{R}^{pqrs} \epsilon_{pq} \epsilon_{rs}, \quad \delta S = \delta (\eta_{e} A) \]

\( \eta_{e} \) is a function of the metric and its deriv’s to a given order

\[ \eta_{e} = \eta_{e} \left( g_{ab}, R_{cdef}, \nabla^{(l)} R_{pqrs} \right) \]
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Case of \( f(R) \) gravities: \( L = f(R, \nabla^{n} R) \)
Also the concept of an effective Newton constant for graviton exchange (effective propagator)

\[
\frac{1}{8\pi G_{\text{eff}}} = E_{R}^{pqrs} \epsilon_{pq} \epsilon_{rs} = \frac{\partial f}{\partial R} (g^{pr} g^{qs} - g^{qr} g^{ps}) \epsilon_{pq} \epsilon_{rs} = \frac{\partial f}{\partial R} = \frac{\eta e}{2\pi}, \quad S = \frac{A}{4 G_{\text{eff}}}
\]
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\[
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For these theories, the different polarizations of the gravitons only enter in the definition of the effective Newton constant through the metric itself.
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\]

For these theories, the different polarizations of the gravitons only enter in the definition of the effective Newton constant through the metric itself.

Final result, for \( f(R) \) gravities:

*the local field equations can be thought of as an equation of state of equilibrium thermodynamics* (as in the GR case)
Jacobson’s argument non-trivially extended to $f(R)$ gravity field equations as EoS of local space-time thermodynamics.

EE, P. Silva, Phys Rev D78, 061501(R) (2008), arXiv:0804.3721v2
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Thanks for your attention
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