Chowla-Selberg and other formulas useful for zeta regularization

EMILIO ELIZALDE
ICE/CSIC & IEEC, UAB, Barcelona

Mathematical Structures in Quantum Systems and applications
Benasque, July 8-14, 2012
The Riemann zeta function as a regularization tool.

General scheme for Linear and Quadratic cases. Truncations.

Spectrum only known Implicitly.

The Chowla-Selberg formula in Number Theory.

The Chowla-Selberg series formula (CS). Nontrivial Extensions (ECS).

Operator Zeta Functions: $\zeta_A$ for $A$ a $\Psi$DO, Det’s.

Dixmier trace, Wodzicki Residue.

Multiplicative Anomaly or Defect.
\[ \zeta(s) = \sum \frac{1}{n^s} \]
\[ Z(s) = \sum_{n=1}^{\infty} n^{-s} \]

\[ Z(0) = -\frac{1}{2} \quad \text{or} \quad 1+1+1+\ldots = -\frac{1}{2} \]

\[ Z(-1) = -\frac{1}{12} \quad \text{or} \quad 1+2+3+\ldots = -\frac{1}{12} \]
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \frac{1}{1 - \frac{1}{p^s}}
\]

The prime number theorem

\[
\pi(x) = \# \{ \text{primes } p \leq x \} \sim \frac{x}{\log x}
\]

\[
\zeta(s) \equiv \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \zeta(1-s)
\]

(Goldfeld + Miller, BAMS '03) (completed \(\zeta\))

(E) Entirety: \(\zeta(s)\) meromorphic c., \(s = 0, 1\) poles

(FE) \(\zeta(s) = \zeta(1-s)\)

(BV) Bounded in vertical strips:

\[
\zeta(s) + \frac{1}{s} + \frac{1}{1-s} \quad \text{bounded} \quad -\infty < a < \text{Re}\, s < b < \infty
\]

Riemann (1859)

Poisson s. f.: \[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)
\]

\[
\hat{f}(r) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i r x} \, dx , \quad f \text{ Schwartz}
\]

1) FE 3

2) \(\hat{f}(r) = \frac{1}{V^2} e^{-\pi r^2/V^2} \rightarrow Jochi id.

\[
\Theta(it) = \frac{1}{V^2} \Theta\left(\frac{t}{V}\right) , \quad \Theta(t) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{\pi i n^2 t}
\]

Dirichlet \(L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}\)

Hamburger
Basic strategies

- **Jacobi’s identity** for the $\theta$–function

$$\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \quad q := e^{i\pi \tau}, \ \tau \in \mathbb{C}$$

$$\theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi \tau} \theta_3 \left( \frac{z}{\tau} \left| -\frac{1}{\tau} \right. \right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi nz), \quad z, t \in \mathbb{C}, \ \text{Re}t > 0$$

- **Higher dimensions:** **Poisson summation formula** (Riemann)

$$\sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \hat{f}(\vec{m})$$

$\hat{f}$ Fourier transform

[Gelbart + Miller, BAMS ’03, Iwaniec, Morgan, ICM ’06]

- **Truncated sums** $\longrightarrow$ asymptotic series
3: EXPLICIT CALCULATIONS

 Epstein zeta functions

\[ \zeta_E = \sum_{\vec{n} \in \mathbb{Z}^d} \varphi(\vec{n})^{-s} \]  

 0 quadratic form

 Barnes zeta functions

\[ \zeta_B = \sum_{\vec{n} \in \mathbb{N}^d} \lambda(\vec{n})^{-s} \]  

 linear

 affine form

coefficients \( \in \mathfrak{B} \)

 Extension:

\[ \zeta_E \rightarrow \varphi + \lambda \text{ affine} \]

\[ \sum_{\vec{n} \in \mathbb{N}^d} \]  

 (truncation)

\[ \zeta_B \rightarrow \zeta_B'(0) \text{ (new formulas)} \]

\[ \sum_{\vec{n} \in \mathbb{Z}^d} \]  

 (by analytic cont.)
Example of the ball:

- Operator

\[ (-\Delta + m^2) \]

on the \( D \)-dim ball \( B^D = \{ x \in \mathbb{R}^D ; |x| \leq R \} \)

with Dirichlet, Neumann or Robin BC

- The zeta function

\[ \zeta(s) = \sum_k \lambda_k^{-s} \]

- Eigenvalues implicitly obtained from

\[ (-\Delta + m^2) \phi_k(x) = \lambda_k \phi_k(x) + BC \]

- In spherical coordinates:

\[ \phi_{l,m,n}(r, \Omega) = r^{1-\frac{D}{2}} J_{l+\frac{D-2}{2}}(w_{l,n} r) Y_{l+\frac{D}{2}}(\Omega) \]

\( J_{l+(D-2)/2} \) Bessel functions

\( Y_{l+D/2} \) hyperspherical harmonics

- Eigenvalues \( w_{l,n} > 0 \) determined through BC

\[ J_{l+\frac{D-2}{2}}(w_{l,n} R) = 0, \text{ for Dirichlet BC} \]
\[ \frac{u}{R} J_{l+\frac{D-2}{2}}(w_{l,n}R) + w_{l,n} J'_{l+\frac{D-2}{2}}(w_{l,n}r) \big|_{r=R} = 0, \text{ for Robin BC} \]

- Now, \( \lambda_{l,n} = w_{l,n}^2 + m^2 \)

\[ \zeta(s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} d_l(D)(w_{l,n}^2 + m^2)^{-s} \]

\( w_{l,n} (> 0) \) is defined as the \( n \)-th root of the \( l \)-th equation, \( d_l(D) = (2l + D - 2) \frac{(l+D-3)!}{l!(D-2)!} \)

- Procedure:
  - Contour integral on the complex plane

\[ \zeta(s) = \sum_{l=0}^{\infty} d_l(D) \int_{\gamma} \frac{dk}{2\pi i} \left( k^2 + m^2 \right)^{-s} \frac{\partial}{\partial k} \ln \Phi_{l+\frac{D-2}{2}}(kR) \]

\( \gamma \) runs counterclockwise and must enclose all the solutions [Ginzburg, Van Kampen, EE + I. Brevik]

- Obtained: [with Bordag, Kirsten, Leseduarte, Vassilievich, ...]
  - Zeta functions
  - Determinants
  - Seeley [heat-kernel] coefficients
The Chowla-Selberg Formula (CS)

The Chowla-Selberg Formula (CS)


1. This paper contains a short account of results whose detailed proofs will be published later.

We define the function $Z(s)$ by

$$Z(s) = \sum' (am^2 + bmn + cn^2)^{-s}$$  \hspace{1cm} (1)

where $s = \sigma + it$ (and $t$, real), $\sigma > 1$, and the summation is for all integers $m, n$ (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further $a$ and $c$ are positive numbers while $b$ is real and subject to $4ac - b^2 = \Delta > 0$.

It is well known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole $s$-plane, and satisfies a functional equation similar to the one satisfied by the Riemann Zeta Function. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

Deuring (Math. Ztschr., 37, 403–413 (1933)) obtained an important formula for $Z(s)$. Deuring’s work led Heilbronn (Quart. J. Maths., Oxford, 5, 150 (1934)) to the proof of the following famous conjecture of Gauss on the class-number of binary quadratic forms with a negative fundamental discriminant: let $h(-\Delta)$ denote the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, then

$$h(-\Delta) \to \infty \quad \text{as} \quad \Delta \to \infty$$  \hspace{1cm} (2)

Again using the ideas of Heilbronn and Deuring, Siegel proved that

$$h(-\Delta) > \Delta^{1/2} - \epsilon \quad [\Delta > \Delta_0(\epsilon)]$$  \hspace{1cm} (3)

which is a great advance on (2).

Our starting point is the formula:

$$Z(s) = 2\zeta(2s)a^{-s} + \frac{2^s a^{s-1} \sqrt{\pi}}{\Gamma(s) \Delta^{s-1/2}} \zeta(2s - 1) \Gamma(s - 1/2) + Q(s)$$  \hspace{1cm} (4)

where

$$Q(s) = \frac{\pi^s \cdot 2^s + 1/2}{a^{1/2} \Gamma(s) \Delta^{s/2 - 1/4}} \sum_{n=1}^{\infty} n^{s - 1/2} \sum_{\sigma_1, \sigma_2} \cos \left( \frac{n \pi b}{a} \right) \phi^{s - 1/2} \exp \left\{ - \frac{\pi n \Delta^{1/2}}{2a} \left( \phi + \phi^{-1} \right) \right\} d\phi$$  \hspace{1cm} (4)
The Chowla-Selberg Formula (CS)


Journal für die reine und angewandte Mathematik

gegründet von A. L. Crelle 1826

fortgeführt von
C. W. Borchardt, K. Weierstrass, L. Kronecker,
L. Fuchs, K. Hensel, L. Schlesinger

gegenwärtig herausgegeben von

Helmut Hasse und Hans Rohrbach

unter Mitwirkung von

W. Brödel, M. Deuring, A. Grothendieck, P. R. Halmos, O. Haupt,
F. Hirzebruch, E. Hopf, M. Kneser, G. Köthe, K. Prachar, H. Reichardt,
P. Roquette, W. Schmeidler, L. Schmetterer, E. Stiefel

Band 227

779

Institut für Reine und Angewandte Mathematik
Technische Hochschule Aachen

Walter de Gruyter & Co.
Berlin 1967
On Epstein's Zeta-function

By Atle Selberg at Princeton (N. J.), and S. Chowla at State College (Pa.)

Introduction

This paper was written in the Spring of 1949, and a résumé appeared in the note: On Epstein's zeta Function (I), Proceedings of the National Academy of Sciences (U. S. A.), 35 (1949), 371–374.

Meanwhile, the following papers which have reference to the Proceedings paper, came to our attention:


§ 1.

We define the function $Z(s)$ by

(1) \[ Z(s) = \sum' (am^2 + bmn + cn^2)^{-s} \]

where $s = \sigma + it$ ($\sigma$ and $t$, real), $\sigma > 1$, and the summation is for all integers $m, n$ (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further $a$ and $c$ are positive numbers while $b$ is real and subject to $4ac - b^2 = \Delta > 0$.

It is well-known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole $s$-plane. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

In 1933, Deuring obtained an important formula for $Z(s)$. Deuring's work led Heilbronn to his proof of a famous conjecture of Gauss on the class number of binary quadratic forms with a negative fundamental discriminant. If $h(-\Delta)$ is the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$; Gauss conjectured that

(2) \[ h(-\Delta) \to \infty \text{ as } \Delta \to \infty. \]
Transforming this we get
\[ \sum_{j=1}^{h} \log \Delta \left( \frac{b_j + i \sqrt{|d|}}{2a_j} \right) = 6 \left\{ h \gamma + \log \prod_{j=1}^{h} \frac{a_j}{|d|^{\frac{3}{2}}} \right\} - \frac{3w}{\pi} \sqrt{|d|} L'_d(1). \]

Inserting here the value (obtained like (58))
\[ L'_d(1) = -\frac{\pi}{\sqrt{|d|}} \sum_{m=1}^{[d]} \left( \frac{d}{m} \right) \log \Gamma \left( \frac{m}{|d|} \right) + \frac{2h \pi (\gamma + \log 2\pi)}{w \sqrt{|d|}} \]
one gets, writing \( \tau_j = \frac{b_j + i \sqrt{|d|}}{2a_j} \),
\[ \prod_{j=1}^{h} \Delta(\tau_j) = \frac{\prod_{j=1}^{h} a_j}{(2\pi |d|)^{3w}} \left\{ \prod_{m=1}^{[d]} \Gamma \left( \frac{m}{|d|} \right) \left( \frac{d}{m} \right)^{3w} \right\}^{\lambda_j} \]

Now let \( \tau = i \frac{K'}{K} \) be a number from the field \( k(\sqrt{d}) \), then from Lemma 3 we get
\[ \frac{\Delta(\tau_0)}{\Delta(\tau)} = \lambda_j, \]
where \( \lambda_j \) are algebraic numbers. Thus (2) gives
\[ \Delta(\tau) = \frac{\lambda'}{\pi^6} \left\{ \prod_{m=1}^{[d]} \Gamma \left( \frac{m}{|d|} \right) \left( \frac{d}{m} \right)^{\frac{3w}{\lambda}} \right\}, \]
where \( \lambda' \) is an algebraic number. Finally we have from (48)
\[ \Delta(\tau) = \left( \frac{2K}{\pi} \right)^{12} \cdot 2^{-4}(kk')^4 = \lambda'' \left( \frac{K}{\pi} \right)^{12}, \]
where \( \lambda'' \) is an algebraic number. This gives, when inserted in (3)
\[ K = \lambda'' \sqrt{\pi} \left\{ \prod_{m=1}^{[d]} \Gamma \left( \frac{m}{|d|} \right) \left( \frac{d}{m} \right)^{\frac{w}{4\pi}} \right\}, \]
which is the desired expression for \( K \) in finite terms.

References (in the order of appearance in the text)
The Chowla-Selberg Formula (CS)

The Chowla-Selberg Formula (CS)


The Chowla-Selberg Formula (CS)

The Chowla-Selberg Formula (CS)


The Chowla-Selberg Formula (CS)


- P. Deligne, *Valeurs de fonctions L et periodes d’integrales*, PSPM 33 (1979) 313-346
Lerch (1897):

\[
\sum_{\lambda=1}^{\lfloor D \rfloor} \left( \frac{D}{\lambda} \right) \log \Gamma \left( \frac{\lambda}{D} \right) = h \log |D| - \frac{h}{3} \log(2\pi) - \sum_{(a,b,c)} \log a
\]

\[
+ \frac{2}{3} \sum_{(a,b,c)} \log \left[ \theta'_1(0|\alpha)\theta'_1(0|\beta) \right]
\]

\(D\) discriminant, \(\theta'_1 \sim \eta^3\)

\(h\) class number of binary quadratic forms \((a, b, c)\)
History

Lerch (1897):

\[
\sum_{\lambda=1}^{\lfloor D \rfloor} \left( \frac{D}{\lambda} \right) \log \Gamma \left( \frac{\lambda}{D} \right) = h \log |D| - \frac{h}{3} \log(2\pi) - \sum_{(a,b,c)} \log a
\]

\[
+ \frac{2}{3} \sum_{(a,b,c)} \log [\theta'_1(0|\alpha)\theta'_1(0|\beta)]
\]

\(D\) discriminant, \(\theta'_1 \sim \eta^3\)

\(h\) class number of binary quadratic forms \((a, b, c)\)

**Eta evaluations** Dedekind eta function for \(\text{Im}(\tau) > 0\)

\[\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q := e^{2\pi i \tau}\]

It is a 24-th root of the discriminant func \(\Delta(\tau)\) of an elliptic curve \(\mathbb{C}/L\) from a lattice \(L = \{a\tau + b \mid a, b \in \mathbb{Z}\}\)

\[\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}\]
The C-S formula gives the value of a product of eta functions.
Properties & Recent Results

The C-S formula gives the value of a product of eta functions

If there is only one form in the class, it yields the value of a single eta function in terms of gamma functions
Properties & Recent Results

The C-S formula gives the value of a product of eta functions.

If there is only one form in the class, it yields the value of a single eta function in terms of gamma functions.

Long series of improvements: Kaneko (90), Nakajima and Taguchi (91), Williams et al. (95)
The C-S formula gives the value of a product of eta functions.

If there is only one form in the class, it yields the value of a single eta function in terms of gamma functions.

Long series of improvements: Kaneko (90), Nakajima and Taguchi (91), Williams et al. (95).

In the last years the C-S formula has been ‘broken’ to isolate the eta functions: Williams, van Poorten, Chapman, Hart.
The C-S formula gives the value of a product of eta functions.

If there is only one form in the class, it yields the value of a single eta function in terms of gamma functions.

Long series of improvements: Kaneko (90), Nakajima and Taguchi (91), Williams et al. (95).

In the last years the C-S formula has been ‘broken’ to isolate the eta functions: Williams, van Poorten, Chapman, Hart.

Properties & Recent Results

- The C-S formula gives the value of a product of eta functions
- If there is only one form in the class, it yields the value of a single eta function in terms of gamma functions
- Long series of improvements: Kaneko (90), Nakajima and Taguchi (91), Williams et al. (95)
- In the last years the C-S formula has been ‘broken’ to isolate the eta functions:
  - Williams, van Poorten, Chapman, Hart
Properties & Recent Results

⇒ The C-S formula gives the value of a product of eta functions

⇒ If there is only one form in the class, it yields the value of a single eta function in terms of gamma functions

⇒ Long series of improvements: Kaneko (90), Nakajima and Taguchi (91), Williams et al. (95)

⇒ In the last years the C-S formula has been ‘broken’ to isolate the eta functions:
  Williams, van Poorten, Chapman, Hart


Properties & Recent Results

⇒ The C-S formula gives the value of a product of eta functions.

⇒ If there is only one form in the class, it yields the value of a single eta function in terms of gamma functions.

⇒ Long series of improvements: Kaneko (90), Nakajima and Taguchi (91), Williams et al. (95).

⇒ In the last years the C-S formula has been ‘broken’ to isolate the eta functions: Williams, van Poorten, Chapman, Hart.

Consider the zeta function \((\text{Re} s > p/2, A > 0, \text{Re} q > 0)\)

\[
\zeta_{A, \vec{c}, q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{n} \in \mathbb{Z}^p} \left[ Q (\vec{n} + \vec{c}) + q \right]^{-s}
\]

Prime: point \(\vec{n} = \vec{0}\) to be excluded from the sum

(inescapable condition when \(c_1 = \cdots = c_p = q = 0\))

\[
Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}
\]
Consider the zeta function \((\text{Re} s > p/2, A > 0, \text{Re} q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{n} \in \mathbb{Z}^p} [Q (\vec{n} + \vec{c}) + q]^{-s}
\]

prime: point \(\vec{n} = \vec{0}\) to be excluded from the sum

(inescapable condition when \(c_1 = \cdots = c_p = q = 0\))

\[Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}\]

Case \(q \neq 0\) \((\text{Re} q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2} \pi^s q^{-s/2+p/4}}{\sqrt{\det A} \Gamma(s)}
\times \sum_{\vec{m} \in \mathbb{Z}_{1/2}^p} \cos(2\pi \vec{m} \cdot \vec{c}) (\vec{m}^T A^{-1} \vec{m})^{s/2-p/4} K_{p/2-s} \left(2\pi \sqrt{2q \vec{m}^T A^{-1} \vec{m}}\right)
\]
Consider the zeta function \((\text{Re} s > p/2, A > 0, \text{Re} q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \sum' \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum' \left[ Q (\vec{n} + \vec{c}) + q \right]^{-s}
\]

**prime:** point \(\vec{n} = \vec{0}\) to be excluded from the sum

(inescapable condition when \(c_1 = \cdots = c_p = q = 0\))

\[Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}\]

**Case** \(q \neq 0 (\text{Re} q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2} \pi^s q^{-s/2+p/4}}{\sqrt{\det A} \Gamma(s)}
\]

\[\times \sum' \cos(2\pi \vec{m} \cdot \vec{c}) \left( \vec{m}^T A^{-1} \vec{m} \right)^{s/2-p/4} K_{p/2-s} \left( 2\pi \sqrt{2q} \vec{m}^T A^{-1} \vec{m} \right) \]

**Pole:** \(s = p/2\)

**Residue:**

\[
\text{Res}_{s=p/2} \zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}
\]
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.

Exhibits singularities (simple poles) of the meromorphic continuation—with the corresponding residua—explicitly.
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane. Exhibits singularities (simple poles) of the meromorphic continuation—with the corresponding residua—explicitly.

Only condition on matrix $A$: corresponds to (non negative) quadratic form, $Q$. Vector $\vec{c}$ arbitrary, while $q$ is (to start) a non-neg constant.
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane

Exhibits singularities (simple poles) of the meromorphic continuation—with the corresponding residua—explicitly

Only condition on matrix $A$: corresponds to (non negative) quadratic form, $Q$. Vector $\vec{c}$ arbitrary, while $q$ is (to start) a non-neg constant

$K_{\nu}$ modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose first non-zero component is positive]
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.

Exhibits singularities (simple poles) of the meromorphic continuation—with the corresponding residua—explicitly.

Only condition on matrix $A$: corresponds to (non negative) quadratic form, $Q$. Vector $\vec{c}$ arbitrary, while $q$ is (to start) a non-neg constant.

$K_{\nu}$ modified Bessel function of the second kind and the subindex $1/2$ in $\mathbb{Z}_{1/2}^p$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$.

[simple criterion: one may select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose first non-zero component is positive]

Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]
QFT in s-t with non-comm toroidal part

- $D$-dim non-commut manifold: $M = \mathbb{R}^{1,d} \otimes \mathbb{T}_\theta^p$, $D = d + p + 1$

- $\mathbb{T}_\theta^p$ a $p$-dim non-commutative torus: $[x_j, x_k] = i\theta \sigma_{jk}$

- $\sigma_{jk}$ a real, nonsingular, antisymmetric matrix of $\pm 1$ entries

- $\theta$ the non-commutative parameter.

[Quresh’s talk]
QFT in s-t with non-comm toroidal part

\[ D\text{-dim non-commut manifold: } M = \mathbb{R}^{1,d} \otimes T^p_\theta, \quad D = d + p + 1 \]

\[ T^p_\theta \text{ a } p\text{-dim non-commutative torus: } [x_j, x_k] = i\theta \sigma_{jk} \]

\[ \sigma_{jk} \text{ a real, nonsingular, antisymmetric matrix of } \pm 1 \text{ entries} \]

\[ \theta \text{ the non-commutative parameter.} \]

Interest recently, in connection with \( M \)-theory & string theory

[Connes,Douglas,Seiberg,Cheung,Chu,Chomerus,Ardalan, ... ]
QFT in s-t with non-comm toroidal part

\[ D \text{-dim non-commut manifold: } M = \mathbb{R}^{1,d} \otimes \mathbb{T}_\theta^p, \quad D = d + p + 1 \]

\[ \mathbb{T}_\theta^p \text{ a } p\text{-dim non-commutative torus: } [x_j, x_k] = i\theta \sigma_{jk} \]

\[ \sigma_{jk} \text{ a real, nonsingular, antisymmetric matrix of \( \pm 1 \) entries} \]

\[ \theta \text{ the non-commutative parameter.} \]

Interest recently, in connection with \( M \text{—theory \\& string theory} \)
[Connes,Douglas,Seiberg,Cheung,Chu,Chomerus,Ardalan, ...]

Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with \# of compact, noncompact, and noncommutative dimensions

\[ \zeta_\alpha(s) = \frac{V \Gamma(s - (d + 1)/2)}{(4\pi)^{(d+1)/2} \Gamma(s)} \sum_{\vec{n} \in \mathbb{Z}^p} \left( Q(\vec{n})^{(d+1)/2-s} [1+\Lambda \theta^2 - 2\alpha Q(\vec{n})^{-\alpha}]^{(d+1)/2-s} \right) \]

\[ \alpha = 2 \text{ bos, } \alpha = 3 \text{ ferm, } \quad V = \text{Vol} (\mathbb{R}^{d+1}) \text{ of non-compact part} \]

\[ Q(\vec{n}) = \sum_{j=1}^{p} a_j n_j^2 \text{ a diag quadratic form, } R_j = a_j^{-1/2} \text{ compactific radii} \]
\( \zeta_\alpha(s) = \frac{2^{s-d} V}{(2\pi)^{(d+1)/2} \Gamma(s)} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - (d + 1)/2)}{l! \Gamma(s + \alpha l - (d + 1)/2)} (-2^{\alpha} \Lambda \theta^{2-2\alpha}) \sum_{j=0}^{p-1} (\det A_j)^{-\frac{1}{2}} \)

\[ \times \left[ \pi^{j/2} a_{p-j}^{-s-\alpha l+(d+j+1)/2} \Gamma(s + \alpha l - (d + j + 1)/2) \zeta_R(2s + 2\alpha l - d - j - 1) \right. \]

\[ \left. + 4\pi^{s+\alpha l-(d+1)/2} a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j} 'n^{(d+j+1)/2-s-\alpha l} \right] \]

\[ \times \left( \vec{m}_j^t A_j^{-1} \vec{m}_j \right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l} \left( 2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j} \right) \]
\[ \zeta_\alpha(s) = \frac{2^{s-d} V}{(2\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - (d + 1)/2)}{l! \Gamma(s + \alpha l - (d + 1)/2)} (-2^{\alpha} \Lambda \theta_{\alpha}^{2-2\alpha})^l \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \]

\times \left[ \pi^{j/2} a_{p-j}^{-s-\alpha l + (d+j+1)/2} \Gamma(s + \alpha l - (d + j + 1)/2) \zeta_R(2s + 2\alpha l - d - j - 1) \right]

+ \left[ 4\pi^{s+\alpha l - (d+1)/2} a_{p-j}^{-(s+\alpha l)/2 - (d+j+1)/4} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j} ' n^{(d+j+1)/2 - s - \alpha l} \right]

\times \left( \vec{m}_j^t A_j^{-1} \vec{m}_j \right)^{(s+\alpha l)/2 - (d+j+1)/4} K_{(d+j+1)/2 - s - \alpha l} \left( 2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j} \right) \]

<table>
<thead>
<tr>
<th>p \ (D)</th>
<th>even</th>
<th>odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>(1a) pole / finite ((l \geq l_1))</td>
<td>(2a) pole / pole</td>
</tr>
<tr>
<td>even</td>
<td>(1b) double pole / pole ((l \geq l_1, l_2))</td>
<td>(2b) pole / double pole ((l \geq l_2))</td>
</tr>
</tbody>
</table>

- General pole structure of \( \zeta_\alpha(s) \), for the possible values of \( D \) and \( p \) being odd or even. Magenta, type of behavior corresponding to lower values of \( l \); behavior in blue corresponds to larger values of \( l \).
After some calculations,

\[
\zeta_\alpha(s) = \frac{V}{(4\pi)^{d+1}/2} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - \frac{d+1}{2})}{l! \Gamma(s)} \left( -\Lambda \theta^{2-2\alpha} \right)^l \zeta_{Q,0,0}(s + \alpha l - \frac{d+1}{2})
\]

for all radii equal to \( R \), with \( I(\vec{n}) = \sum_{j=1}^{p} n_j^2 \),

\[
\zeta_\alpha(s) = \frac{V}{(4\pi)^{d+1}/2 R^{d+1-2s}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - \frac{d+1}{2})}{l! \Gamma(s)} \left( -\Lambda \theta^{2-2\alpha} \right)^l \zeta_E(s + \alpha l - \frac{d+1}{2})
\]

where we use the notation \( \zeta_E(s) := \zeta_{I,0,0}(s) \)
e.g., the Epstein zeta function for the standard quadratic form
After some calculations,

\[
\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - \frac{d+1}{2})}{l! \Gamma(s)} (-\Lambda \theta^{2-2\alpha})^l \zeta_Q,\bar{0},0(s + \alpha l - \frac{d + 1}{2})
\]

for all radii equal to \( R \), with \( I(\vec{n}) = \sum_{j=1}^{p} n_j^2 \),

\[
\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2} R^{d+1-2s}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - \frac{d+1}{2})}{l! \Gamma(s)} (-\Lambda \theta^{2-2\alpha})^l \zeta_E(s + \alpha l - \frac{d + 1}{2})
\]

where we use the notation \( \zeta_E(s) := \zeta_{I,\bar{0},0}(s) \)

e.g., the Epstein zeta function for the standard quadratic form

Rich pole structure: pole of Epstein zf at
\( s = p/2 - \alpha k + (d + 1)/2 = D/2 - \alpha k \), combined with
poles of \( \Gamma \), yields a rich pattern of singul for \( \zeta_\alpha(s) \)
After some calculations,

\[
\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - \frac{d+1}{2})}{l! \Gamma(s)} \left(-\Lambda \theta^{2-2\alpha}\right)^l \zeta_{Q,\vec{0},0}(s + \alpha l - \frac{d + 1}{2})
\]

for all radii equal to \( R \), with \( I(\vec{n}) = \sum_{j=1}^{p} n_j^2 \),

\[
\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \frac{1}{R^{d+1-2s}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - \frac{d+1}{2})}{l! \Gamma(s)} \left(-\Lambda \theta^{2-2\alpha}\right)^l \zeta_{E}(s + \alpha l - \frac{d + 1}{2})
\]

where we use the notation \( \zeta_{E}(s) := \zeta_{I,\vec{0},0}(s) \),

e.g., the Epstein zeta function for the standard quadratic form

**Rich pole structure:** pole of Epstein zf at \( s = p/2 - \alpha k + (d + 1)/2 = D/2 - \alpha k \), combined with poles of \( \Gamma \), yields a rich pattern of singul for \( \zeta_\alpha(s) \)

**Classify** the different possible cases according to the values of \( d \) and \( D = d + p + 1 \). We obtain, at \( s = 0 \):
For $d = 2k$

\[
\begin{aligned}
\text{if } D \neq \frac{2}{2\alpha} & \implies \zeta_{\alpha}(0) = 0 \\
\text{if } D = \frac{2}{2\alpha} & \implies \zeta_{\alpha}(0) = \text{finite}
\end{aligned}
\]

For $d = 2k - 1$

\[
\begin{aligned}
\text{if } D \neq \frac{2}{2\alpha} & \begin{cases}
\text{finite, for } l \leq k \\
0, & \text{for } l > k
\end{cases} \implies \zeta_{\alpha}(0) = \text{finite} \\
\text{if } D = 2\alpha l & \begin{cases}
\text{pole, for } l \leq k \\
\text{finite, for } l > k
\end{cases} \implies \zeta_{\alpha}(0) = \text{pole}
\end{aligned}
\]

- Pole structure of the zeta function $\zeta_{\alpha}(s)$, at $s = 0$, according to the different possible values of $d$ and $D$ ($\frac{2}{2\alpha}$ means multiple of $2\alpha$)
For $d = 2k$
\[
\begin{cases}
\text{if } D \neq \overline{2\alpha} & \implies \zeta_\alpha(0) = 0 \\
\text{if } D = \overline{2\alpha} & \implies \zeta_\alpha(0) = \text{finite}
\end{cases}
\]

\[
\begin{cases}
\text{if } D \neq \overline{2\alpha} & \\
\begin{cases}
\text{finite, for } l \leq k \\
0, \text{ for } l > k
\end{cases} & \implies \zeta_\alpha(0) = \text{finite}
\end{cases}
\]

For $d = 2k - 1$
\[
\begin{cases}
\text{if } D = 2\alpha l & \\
\begin{cases}
pole, \text{ for } l \leq k \\
\text{finite, for } l > k
\end{cases} & \implies \zeta_\alpha(0) = \text{pole}
\end{cases}
\]

– Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of $d$ and $D$ ($\overline{2\alpha}$ means multiple of $2\alpha$)

$\implies$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$, & specific pole structure
A \textit{ΨDO} of order $m$ on $M_n$ manifold

Symbol of $A$: $a(x, \xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices $\alpha, \beta$ there exists a constant $C_{\alpha,\beta}$ so that

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}$$
Pseudodifferential Operator ($\Psi$DO)

- **A $\Psi$DO of order $m$**
  - $M_n$ manifold

- **Symbol of $A$:** $a(x, \xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices $\alpha, \beta$ there exists a constant $C_{\alpha, \beta}$ so that
  \[
  \left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|}
  \]

- **Definition of $A$ (in the distribution sense)**
  \[
  Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x, \xi) \hat{f}(\xi) \, d\xi
  \]

  - $\hat{f}$ is the Fourier transform of $f$
  - $f$ is a smooth function
    \[
    f \in S = \{ f \in C^\infty(\mathbb{R}^n); \sup_x |x^\beta \partial^\alpha f(x)| < \infty, \ \forall \alpha, \beta \in \mathbb{N}^n \}
    \]

  - $S'$ space of tempered distributions
ψDOs are useful tools

The symbol of a ψDO has the form:

\[ a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots + a_{m-j}(x, \xi) + \cdots \]

being \( a_k(x, \xi) = b_k(x) \xi^k \)

\( a(x, \xi) \) is said to be elliptic if it is invertible for large \( |\xi| \) and if there exists a constant \( C \) such that \[ |a(x, \xi)^{-1}| \leq C(1 + |\xi|)^{-m}, \text{ for } |\xi| \geq C \]

– An elliptic ψDO is one with an elliptic symbol
**ΨDOs are useful tools**

The symbol of a ΨDO has the form:

$$a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots + a_m-j(x, \xi) + \cdots$$

being $a_k(x, \xi) = b_k(x) \xi^k$

$a(x, \xi)$ is said to be **elliptic** if it is invertible for large $|\xi|$ and if there exists a constant $C$ such that $|a(x, \xi)^{-1}| \leq C(1 + |\xi|)^{-m}$, for $|\xi| \geq C$

---

- An elliptic ΨDO is one with an elliptic symbol

---

ΨDOs are basic tools both in Mathematics & in Physics

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through $\mu$localization (the most important step towards the understanding of linear PDEs since the invention of distributions) [K Fredenhagen, R Brunetti, ... R Wald ’06, R Haag EPJH35 ’10]
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of $E$

3. $E$, $n$-dim vector bundle over $M$

4. $M$ closed $n$-dim manifold
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$
2. $A$ acts on the space of smooth sections of
3. $E$, $n$-dim vector bundle over
4. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of $A$, $s_0 = \dim M / \text{ord } A$ abscissa of converg of $\zeta_A(s)$
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of

3. $E$, $n$-dim vector bundle over

4. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of $A$, $s_0 = \dim M/\text{ord } A$ abscissa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$

(the Agmon-Nirenberg condition)
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$
2. $A$ acts on the space of smooth sections of $E$, $n$-dim vector bundle over $M$
3. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:
$$\zeta_A(s) = \text{tr} A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re} s > \frac{n}{m} := s_0$$
\{\lambda_j\} ordered spect of $A$, $s_0 = \text{dim} M/\text{ord} A$ abscessa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg} \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec} A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$
2. $A$ acts on the space of smooth sections of
3. $E$, $n$-dim vector bundle over
4. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:
$$
\zeta_A(s) = \text{tr} A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0
$$
\{\lambda_j\} ordered spect of $A$, $s_0 = \dim M/\text{ord } A$ abscessa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$

(d) The only possible singularities of $\zeta_A(s)$ are poles at
$$
 s_j = (n - j)/m, \quad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots
$$
Definition of Determinant

\[ H \psi \text{DO operator} \{ \varphi_i, \lambda_i \} \text{ spectral decomposition} \]
Definition of Determinant

\[ H \Psi \text{DO operator} \quad \{ \phi_i, \lambda_i \} \quad \text{spectral decomposition} \]

\[ \prod_{i \in I} \lambda_i \quad \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]
Definition of Determinant

\[ H \Psi \text{DO operator} \quad \{\varphi_i, \lambda_i\} \quad \text{spectral decomposition} \]

\[ \prod_{i \in I} \lambda_i \quad ?! \quad \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]

Riemann zeta func: \[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re} \ s > 1 \quad (\& \text{analytic cont}) \]

Definition: zeta function of \( H \)

\[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s} \]

As Mellin transform: \[ \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{tr} \ e^{-tH}, \quad \text{Re} \ s > s_0 \]

Derivative: \[ \zeta_H'(0) = -\sum_{i \in I} \ln \lambda_i \]
Definition of Determinant

\[ H \] DO operator \{\varphi_i, \lambda_i\} spectral decomposition

\[ \prod_{i \in I} \lambda_i \quad \text{?!} \quad \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]

Riemann zeta func: \[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \ Re\ s > 1 \quad (\& \ analytic\ cont) \]

Definition: zeta function of \[ H \]

\[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s} \]

As Mellin transform: \[ \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \text{tr} \ e^{-tH}, \ Re s > s_0 \]

Derivative: \[ \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \]

Determinant: Ray & Singer, ’67

\[ \det \zeta H = \exp [-\zeta'_H(0)] \]
**Definition of Determinant**

\[ H \] \[ \Psi \text{DO operator} \] \[ \{ \varphi_i, \lambda_i \} \] \[ \text{spectral decomposition} \]

\[ \prod_{i \in I} \lambda_i \ ?! \]

\[ \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]

Riemann zeta func: \[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \; Re\ s > 1 \] (& analytic cont)

Definition: zeta function of \( H \)

\[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \; H^{-s} \]

As Mellin transform: \[ \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \; t^{s-1} \text{tr} \; e^{-tH}, \; Re_s > s_0 \]

Derivative: \[ \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \]

Determinant: Ray & Singer, ’67

\[ \det \zeta H = \exp \left[ -\zeta'_H(0) \right] \]

Weierstrass def: subtract leading behavior of \( \lambda_i \) in \( i \), as \( i \to \infty \), until series \( \sum_{i \in I} \ln \lambda_i \) converges \( \implies \) non-local counterterms !!
**Definition of Determinant**

\[ H \text{ DO operator} \quad \{\varphi_i, \lambda_i\} \quad \text{spectral decomposition} \]

\[ \prod_{i \in I} \lambda_i \quad \text{?!} \]

\[ \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]

Riemann zeta func: \[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re} \, s > 1 \quad (\& \text{analytic cont}) \]

Definition: zeta function of \[ H \]

\[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \, H^{-s} \]

As Mellin transform: \[ \zeta_H(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \, t^{s-1} \text{tr} \, e^{-tH}, \quad \text{Re} \, s > s_0 \]

Derivative: \[ \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \]

Determinant: Ray & Singer, ’67

\[ \det_\zeta H = \exp \left[ -\zeta'_H(0) \right] \]

Weierstrass def: subtract leading behavior of \[ \lambda_i \] in \[ i \], as \[ i \to \infty \], until series \[ \sum_{i \in I} \ln \lambda_i \] converges \[ \implies \text{non-local counterterms} !! \]

C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...
**Properties**

- The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut.

- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex $s$-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C}\setminus\mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik].

- Asymptotic expansion for the heat kernel:

\[
\text{tr } e^{-tA} = \sum_{\lambda \in \text{Spec } A} e^{-t\lambda}
\]

\[
\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0
\]

\[
\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \text{Res}_{s=s_j} \zeta_A(s), \quad s_j \notin \mathbb{N}
\]

\[
\alpha_j(A) = \frac{(-1)^k}{k!} \left[ \text{PP } \zeta_A(-k) + \psi(k + 1) \text{Res}_{s=-k} \zeta_A(s) \right],
\]

\[
\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\setminus\{0\}
\]

\[
s_j = -k, \quad k \in \mathbb{N}
\]

\[
\text{PP } \phi := \lim_{s \to p} \left[ \phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]
\]
The Dixmier Trace

In order to write down an action in operator language one needs a functional that replaces integration.
The Dixmier Trace

In order to write down an action in operator language one needs a functional that replaces integration.

For the Yang-Mills theory this is the Dixmier trace.
The Dixmier Trace

In order to write down an action in operator language one needs a functional that replaces integration.

For the Yang-Mills theory this is the Dixmier trace.

It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators $T$ such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \quad \mu_0 \geq \mu_1 \geq \cdots$$
The Dixmier Trace

In order to write down an action in operator language one needs a functional that replaces integration.

For the Yang-Mills theory this is the Dixmier trace.

It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators $T$ such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = O(\log N), \quad \mu_0 \geq \mu_1 \geq \cdots$$

Definition of the Dixmier trace of $T$:

$$\text{Dtr } T = \lim_{N \to \infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in $N$ are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]
The Dixmier Trace

In order to write down an action in operator language one needs a functional that replaces integration.

For the Yang-Mills theory this is the Dixmier trace.

It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators $T$ such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \quad \mu_0 \geq \mu_1 \geq \cdots$$

Definition of the Dixmier trace of $T$:

$$\text{Dtr } T = \lim_{N \to \infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in $N$ are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator $T^{-1}$ at $s = 1$ [Connes]

$$\text{Dtr } T = \lim_{s \to 1^+} (s - 1) \zeta_{T^{-1}}(s)$$
The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to \( \Psi DOs \) which are not in \( \mathcal{L}^{(1,\infty)} \).

- Only trace one can define in the algebra of \( \Psi DOs \) (up to multipl const).

- Definition: \( \text{res } A = 2 \text{ Res}_{s=0} \text{tr } (A\Delta^{-s}), \ \Delta \text{ Laplacian} \)

- Satisfies the trace condition: \( \text{res } (AB) = \text{res } (BA) \)

- Important!: it can be expressed as an integral (local form)

\[
\text{res } A = \int_{S^*M} \text{tr } a_{-n}(x, \xi) \, d\xi
\]

with \( S^*M \subset T^*M \) the co-sphere bundle on \( M \) (some authors put a coefficient in front of the integral: Adler-Manin residue).

- If \( \dim M = n = -\text{ord } A \) (\( M \) compact Riemann, \( A \) elliptic, \( n \in \mathbb{N} \)) it coincides with the Dixmier trace, and \( \text{Res}_{s=1} \zeta_A(s) = \frac{1}{n} \text{res } A^{-1} \)

- The Wodzicki residue makes sense for \( \Psi DOs \) of arbitrary order. Even if the symbols \( a_j(x, \xi), j < m \), are not coordinate invariant, the integral is, and defines a trace.
Singularities of $\zeta_A$

A complete determination of the meromorphic structure of some zeta functions in the complex plane can be also obtained by means of the Dixmier trace and the Wodzicki residue.

Missing for full descript of the singularities: residua of all poles.

As for the regular part of the analytic continuation: specific methods have to be used (see later).

**Proposition.** Under the conditions of existence of the zeta function of $A$, given above, and being the symbol $a(x, \xi)$ of the operator $A$ analytic in $\xi^{-1}$ at $\xi^{-1} = 0$:

$$\text{Res}_{s=s_k} \zeta_A(s) = \frac{1}{m} \text{res} \ A^{-s_k} = \frac{1}{m} \int_{S^* M} \text{tr} \ a^{-s_k}_n(x, \xi) \, dn^{-1} \xi$$

**Proof.** The homog component of degree $-n$ of the corresp power of the principal symbol of $A$ is obtained by the appropriate derivative of a power of the symbol with respect to $\xi^{-1}$ at $\xi^{-1} = 0$:

$$a^{-s_k}_n(x, \xi) = \left( \frac{\partial}{\partial \xi^{-1}} \right)^k \left[ \xi^{n-k} a(k-n)/m(x, \xi) \right]_{\xi^{-1}=0} \xi^{-n}$$
Given $A$, $B$, and $AB \psi$ DOs, even if $\zeta_A$, $\zeta_B$, and $\zeta_{AB}$ exist, it turns out that, in general,

$$\det_{\zeta}(AB) \neq \det_{\zeta}A \det_{\zeta}B$$
Multipl or N-Comm Anomaly, or Defect

- Given $A$, $B$, and $AB$ DOs, even if $\zeta_A$, $\zeta_B$, and $\zeta_{AB}$ exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$$

- The multiplicative (or noncommutative) anomaly (defect) is defined as

$$\delta(A, B) = \ln \left[ \frac{\det_\zeta(AB)}{\det_\zeta A \det_\zeta B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$
Multipl or N-Comm Anomaly, or Defect

- Given $A$, $B$, and $AB$ ψDOs, even if $ζ_A$, $ζ_B$, and $ζ_{AB}$ exist, it turns out that, in general,
  \[ \text{det}_ζ(AB) \neq \text{det}_ζ A \text{ det}_ζ B \]

- The multiplicative (or noncommutative) anomaly (defect) is defined as
  \[ δ(A, B) = \ln \left[ \frac{\text{det}_ζ(AB)}{\text{det}_ζ A \text{ det}_ζ B} \right] = -ζ'_{AB}(0) + ζ'_A(0) + ζ'_B(0) \]

- Wodzicki formula
  \[ δ(A, B) = \frac{\text{res} \left\{ [\ln σ(A, B)]^2 \right\}}{2 \text{ ord } A \text{ ord } B (\text{ord } A + \text{ord } B)} \]

  where
  \[ σ(A, B) = A^{\text{ord } B} B^{−\text{ord } A} \]
Consequences of the Multipl Anomaly

In the path integral formulation

\[ \int [d\Phi] \exp \left\{ - \int d^D x \left[ \Phi^\dagger(x)(\quad) \Phi(x) + \cdots \right] \right\} \]

Gaussian integration:

\[ \rightarrow \quad \det (\quad) \]

\[
\begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{pmatrix}
\quad \rightarrow \quad
\begin{pmatrix}
A \\
B
\end{pmatrix}
\]

\[ \det(AB) \quad \text{or} \quad \det A \cdot \det B \quad ? \]

In a situation where a superselection rule exists, \( AB \) has no sense (much less its determinant):

\[ \Rightarrow \quad \det A \cdot \det B \]

But if diagonal form obtained after change of basis (diag. process), the preserved quantity is:

\[ \Rightarrow \quad \det(AB) \]