Zeta Functions, vacuum fluctuations and Cosmology

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Cosmology and the Quantum Vacuum V (CQVV 2018)

CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

Sep 02 -- Sep 08, 2018
Zero point energy

QFT vacuum to vacuum transition: \( \langle 0 | H | 0 \rangle \)

Spectrum, normal ordering (harm oscil): 

\[
H = \left( n + \frac{1}{2} \right) \lambda_n \ a_n \ a_n^\dagger
\]

\[
\langle 0 | H | 0 \rangle = \frac{\hbar c}{2} \sum_n \lambda_n = \frac{1}{2} \text{tr} \ H
\]

gives \( \infty \) physical meaning?

Regularization + Renormalization (cut-off, dim, \( \zeta \))

Even then: Has the final value real sense?
Effects of the Quantum Vacuum

a) Negligible: Sonoluminiscence, Schwinger \( \sim 10^{-5} \)
b) Important: Wetting He3 – alcali \( \sim 30\% \)
c) Incredibly big: Cosmological constant \( \sim 10^{120} \)
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]
Fundamental, $n = 1$
\[ \lambda_1 = 2L \]

$2^{nd}$ harmonic, $n = 2$
\[ \lambda_2 = L \]

$3^{rd}$ harmonic, $n = 3$
\[ \lambda_3 = \frac{2}{3}L \]

$4^{th}$ harmonic, $n = 4$
\[ \lambda_4 = \frac{1}{2}L \]

$5^{th}$ harmonic, $n = 5$
\[ \lambda_5 = \frac{2}{5}L \]
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \]

\[ \zeta(0) = -\frac{1}{2} \]

\[ \zeta(-1) = -\frac{1}{12} \]

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty \]

\[ 1 + 1 + 1 + \cdots = -\frac{1}{2} \]

\[ 1 + 2 + 3 + \cdots = -\frac{1}{12} \]

F Yndurain, A Slavnov
"As everybody knows ..."
\[ \xi(s) = \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) \]

\[ \xi(1 - s) = \xi(s) \]

\[ \zeta \left( \frac{1}{2} - s' \right) = \xi \left( \frac{1}{2} + s' \right) \]

Real on the critical line

\[ \zeta \left( \frac{1}{2} + it \right) = Z(t) \ e^{-i \Theta(t)} \]

Hardy, Riemann-Siegel
A proof of the Riemann hypothesis on zeros of $\zeta$-function

Vladimir Ryazanov
September 3, 2018

Abstract

Applying the known Nyman–Beurling criterion, it is proved the Riemann hypothesis on zeros of $\zeta$-function.

1 Introduction

In his famous presentation at the International Congress of Mathematicians held in Paris in 1900, David Hilbert included the Riemann Hypothesis as number 8 in his list of 23 challenging problems published later. After over 100 years, it is one of the few on that list that have not been solved. At present many mathematicians consider it the most important unsolved problem in mathematics.

Recall that, exactly one hundred years later, the Clay Mathematics Institute has published a list of 7 unsolved problems for the 21st century, including 6 unresolved problems from the Hilbert list, offering a reward of one million dollars for a solution to any of these problems.

One of them is the Riemann hypothesis, i.e. a conjecture that the so-called Riemann zeta function has as its zeros only complex numbers with real part 1/2 in addition to its trivial zeros at the negative even integers. It was proposed by Bernhard Riemann in his 1859 paper [29]. The Riemann zeta function plays a great role in analytic number theory and applications in physics, probability theory and applied statistics.

2 The Nyman–Beurling criterion

Let us recall the contents of the paper [10] of the known Swedish mathematician Arne Beurling. Denote by $\{\tau\}$ the fractional part $\tau - [\tau]$ of a real number $\tau$ where $[\tau]$ is the greatest integer that is less or equal to $\tau$. Denote also by $B$ the collection of all functions $\varphi : (0, 1) \to \mathbb{R}$ of the form

$$\varphi(t) = \sum_{k=1}^{N} c_k \left\{ \frac{\theta_k}{t} \right\}, \quad c_k \in \mathbb{R}, \theta_k \in (0, 1], \quad k = 1, \ldots, N, \quad \sum_{k=1}^{N} c_k \theta_k = 0 \quad (2.1)$$

Now, let $B_p$ be the closure of $B$ in $L_p = L_p((0, 1), 1 < p < \infty$. It is shown in [10] that $B_p = L_p$ if and only if the function $f(t) \equiv 1$, $t \in (0, 1)$, is in $B_p$. Moreover, it is shown in [10] that the Riemann zeta function has no zeros in $\Re \sigma > 1/p$ if and only if $B_p = L_p$. In particular, by Remark 1 we have from here the following consequence.

Theorem A. The Riemann hypothesis is true if and only if the function $f(t) \equiv 1$, $t \in (0, 1)$, can be approximated in $L_2$ by a sequence from $B$.

Theorem A was first proved in the thesis [28] of Bertil Nyman (1950). Recall also that Beurling was his advisor. The paper [10] first (1955) represented and generalized his result. Later on, the Nyman–Beurling criterion was reproved and generalized in many different ways, as well as, the approach was applied for the research of the problem on the distribution of zeros of the Riemann zeta function, see e.g. [1] [9], [12] [14], [17], [22] [27], [30], [32] [35].

It is impossible
Operator Zeta F’s in $M\Phi$: Origins

The Riemann zeta function $\zeta(s)$ is a function of a complex variable, $s$. To define it, one starts with the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges for all complex values of $s$ with real $\text{Re } s > 1$, and then defines $\zeta(s)$ as the analytic continuation, to the whole complex $s$–plane, of the function given, $\text{Re } s > 1$, by the sum of the preceding series.

Leonhard Euler already considered the above series in 1740, but for positive integer values of $s$, and later Chebyshev extended the definition to $\text{Re } s > 1$.


Did much of the earlier work, by establishing the convergence and equivalence of series regularized with the heat kernel and zeta function regularization methods.

G H Hardy, Divergent Series (Clarendon Press, Oxford, 1949)

Srinivasa I Ramanujan had found for himself the functional equation of the zeta function.

Torsten Carleman, “Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes" (French), 8. Skand Mat-Kongr, 34-44 (1935)

Zeta function encoding the eigenvalues of the Laplacian of a compact Riemannian manifold for the case of a compact region of the plane.

Extended this to **elliptic pseudo-differential** operators $A$ on compact Riemannian manifolds. So for such operators one can define the **determinant** using zeta function regularization


Used this to define the **determinant** of a positive self-adjoint operator $A$ (the Laplacian of a Riemannian manifold in their application) with eigenvalues $a_1, a_2, ....$, and in this case the zeta function is formally the **trace**

$$\zeta_A(s) = \text{Tr} (A)^{-s}$$

the method defines the possibly divergent infinite product

$$\prod_{n=1}^{\infty} a_n = \exp[-\zeta_A'(0)]$$
Abstract

The effective Lagrangian and vacuum energy-momentum tensor $< T^{\mu\nu} >$ due to a scalar field in a de Sitter space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2 + \xi R + m^2)\varphi = 0$. If $\xi = 1/6$ and $m = 0$, the renormalized $< T^{\mu\nu} >$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1}$, where $a$ is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.
Effective Lagrangian and energy-momentum tensor in de Sitter space

J. S. Dowker and Raymond Critchley

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(Received 29 October 1975)

The effective Lagrangian and vacuum energy-momentum tensor \( \langle T^{\mu \nu} \rangle \) due to a scalar field in a de Sitter-space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form \((\Box + \xi R + m^2)\varphi = 0\). If \(\xi = 1/6\) and \(m = 0\), the renormalized \(\langle T^{\mu \nu} \rangle\) equals \(g^{\mu \nu}(960m^2a^4)^{-1}\), where \(a\) is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

I. INTRODUCTION

In a previous paper\(^1\) (to be referred to as I) the effective Lagrangian \(\mathcal{L}^{(1)}\) due to single-loop diagrams of a scalar particle in de Sitter space was computed. It was shown to be real and was evaluated as a principal-part integral. The regularization method used was the proper-time one due to Schwinger\(^2\) and others. We now wish to consider the same problem but using different techniques. In particular, we wish to make contact with the work of Candelas and Raine,\(^3\) who first discussed the same problem using dimensional regularization.

Some properties of the various regularizations as applied to the calculation of the vacuum expectation value of the energy-momentum tensor have been contrasted by DeWitt.\(^4\) We wish to pursue some of these questions within the context of a definite situation.

II. GENERAL FORMULAS: REGULARIZATION METHODS

We use exactly the notation of I, which is more or less standard, and begin with the expression for \(\mathcal{L}^{(1)}\) in terms of the quantum-mechanical propagator, \(K(x'', x', \tau)\),

\[
\mathcal{L}^{(1)}(x') = -\frac{1}{2} i \lim_{x'' \to x'} \int_0^\infty d\tau \tau^{-1} K(x'', x', \tau) e^{-i \tau^2} + X(x').
\]  

(1)

There are two points regarding this expression which need some further discussion. Firstly, if we adopt the proper-time regularization method so that the infinities appear only when the \(\tau\) integration, which is the final operation, is performed, then we can take the coincidence limit, \(x'' = x'\), through into the integrand. Further, since the regularized expression is continuous across the light cone, it does not matter how we let \(x''\) approach \(x'\). Secondly, the term \(X\) does not have to be a constant, but it should integrate to give a metric-independent contribution to the total action, \(\mathcal{W}^{(1)}\).

The Schwinger-DeWitt procedure is to derive an expression for \(K\), either in closed form or as a general expansion to powers of \(\tau\), then to effect the coincidence limit in (1), and finally to perform the \(\tau\) integration. This was the approach adopted in I. We proceed now to give a few more details. We assume that we are working on a Riemannian space, \(\mathcal{M}\), of dimension \(d\). The coincidence limit \(K(x, x, \tau)\) can be expanded,

\[
K(x, x, \tau) = i (4\pi i \tau)^{-d/2} \sum_{n=0}^\infty a_n(x)(i \tau)^n,
\]

(2)

where the \(a_n\) are scalars constructed from the curvature tensor on \(\mathcal{M}\) and whose functional form is independent of \(d\). The manifold \(\mathcal{M}\) must not have boundaries, otherwise other terms appear in the expansion.

The expansion (2) is substituted into (1) to yield

\[
\mathcal{L}^{(1)}(x) = \frac{1}{2} i (4\pi)^{-d/2} \sum_{n} a_n(x) \int_0^\infty (i \tau)^{n-d/2-1} e^{-i \tau^2} d\tau.
\]

(3)

The infinite terms are those for which \(n \leq d/2\) (for \(d\) even) or \(n \leq (d - 1)/2\) (for \(d\) odd). For \(d = 4\), e.g., space-time, there are three infinite terms. These terms are removed by renormalization; the details are given by DeWitt.\(^4\)

Another popular regularization technique is dimensional regularization.\(^6\) In this method the dimension, \(d\), is considered to be complex and all expressions are defined in a region of the \(d\) plane where they converge. The infinities appear when an analytic continuation to \(d = 4\) is performed to regain the physical quantities. This idea was originally developed for use in flat-space (i.e., Lorentz-invariant) situations for the momentum.

This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a **generalized zeta function from the eigenvalues of the differential operator** that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique **agrees with dimensional regularization** where one generalises to $n$ dimensions by adding extra flat dims. The generalized zeta function can be expressed as a **Mellin transform of the kernel of the heat equation** which describes diffusion over the four dimensional spacetime manifold in a fifth dimension of parameter time. Using the **asymptotic expansion for the heat kernel**, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This EM tensor has an anomalous trace.
Zeta Function Regularization of Path Integrals in Curved Spacetime

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Abstract. This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to $n$ dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fifth dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This energy momentum tensor has an anomalous trace.

1. Introduction

The purpose of this paper is to describe a technique for obtaining finite values to path integrals for fields (including the gravitational field) on a curved spacetime background or, equivalently, for evaluating the determinants of differential operators such as the four-dimensional Laplacian or D'Alembertian. One forms a generalised zeta function from the eigenvalues $\lambda_n$ of the operator

$$\zeta(s) = \sum_n \lambda_n^{-s}.$$  \hspace{1cm} (1.1)

In four dimensions this converges for $\text{Re}(s) > 2$ and can be analytically extended to a meromorphic function with poles only at $s = 2$ and $s = 1$. It is regular at $s = 0$. The derivative at $s=0$ is formally equal to $-\sum_n \log \lambda_n$. Thus one can define the determinant of the operator to be $\exp(-d\zeta/ds)|_{s=0}$. 
Basic strategies

- **Jacobi’s identity** for the $\theta$–function

  \[ \theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \quad q := e^{i\pi \tau}, \ \tau \in \mathbb{C} \]

  \[ \theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi \tau} \theta_3 \left( \frac{z}{\tau} | -\frac{1}{\tau} \right) \]

- Equivalently:

  \[ \sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi nz), \quad z, t \in \mathbb{C}, \ \text{Re} t > 0 \]

- Higher dimensions: **Poisson summ formula** (Riemann)

  \[ \sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \tilde{f}(\vec{m}) \]

  \[ \tilde{f} \quad \text{Fourier transform} \]

  [Gelbart + Miller, BAMS ’03, Iwaniec, Morgan, ICM ’06]

- Truncated sums $\longrightarrow$ asymptotic series
3: EXPLICIT CALCULATIONS

\[ Z_E = \sum_{\vec{n} \in \mathbb{Z}^d} \varphi(\vec{n})^{-s} \quad \text{(quadratic)} \]

\[ Z_B = \sum_{\vec{n} \in \mathbb{N}^d} L(\vec{n})^{-s} \quad \text{(linear)} \]

**Barnes zeta function,\]**

\[ \text{(coefficients } \in \mathbb{Q} \text{)} \]

**Extension:**

\[ Z_E \rightarrow \end{equation} \[ \sum_{\vec{n} \in \mathbb{N}^d} \]

\[ Z_B \rightarrow Z_B'(0) \quad \text{(new formula)} \]

\[ \sum_{\vec{n} \in \mathbb{Z}^d} \quad \text{(by analytic continuation)} \]
Example of the ball:

- Operator

\[ (-\Delta + m^2) \]

on the \( D \)-dim ball \( B^D = \{ x \in \mathbb{R}^D; |x| \leq R \} \)

with Dirichlet, Neumann or Robin BC.

- The zeta function

\[ \zeta(s) = \sum_k \lambda_k^{-s} \]

- Eigenvalues implicitly obtained from

\[ (-\Delta + m^2)\phi_k(x) = \lambda_k \phi_k(x) + BC \]

- In spherical coordinates:

\[ \phi_{l,m,n}(r, \Omega) = r^{1-\frac{D}{2}} J_{l+\frac{D-2}{2}}(w_{l,n}r)Y_{l+\frac{D}{2}}(\Omega) \]

\( J_{l+(D-2)/2} \) Bessel functions

\( Y_{l+D/2} \) hyperspherical harmonics

- Eigenvalues \( w_{l,n} (> 0) \) determined through BC

\[ J_{l+\frac{D-2}{2}}(w_{l,n}R) = 0, \quad \text{for Dirichlet BC} \]
\[ \frac{u}{R} J_{l+\frac{D-2}{2}}(w_{l,n} R) + w_{l,n} J'_{l+\frac{D-2}{2}}(w_{l,n} r) \big|_{r=R} = 0, \text{ for Robin BC} \]

- Now, \( \lambda_{l,n} = w_{l,n}^2 + m^2 \)

\[ \zeta(s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} d_l(D)(w_{l,n}^2 + m^2)^{-s} \]

\( w_{l,n} (> 0) \) is defined as the n-th root of the l-th equation, \( d_l(D) = (2l + D - 2) \frac{(l+D-3)!}{l! (D-2)!} \)

- Procedure:
  - Contour integral on the complex plane

\[ \zeta(s) = \sum_{l=0}^{\infty} d_l(D) \int_{\gamma} \frac{dk}{2\pi i} \frac{k^2 + m^2}{(k^2 + m^2)^{-s}} \frac{\partial}{\partial k} \ln \Phi_{l+\frac{D-2}{2}}(kR) \]

\( \gamma \) runs counterclockwise and must enclose all the solutions [Ginzburg, Van Kampen, EE + I. Brevik]

- Obtained: [with Bordag, Kirsten, Leseduarte, Vassilievich, ...]
  - Zeta functions
  - Determinants
  - Seeley [heat-kernel] coefficients
Zeta functions on tori using contour integration

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A new, seemingly useful presentation of zeta functions on complex tori is derived by using contour integration. It is shown to agree with the one obtained by using the Chowla–Selberg series formula, for which an alternative proof is thereby given. In addition, a new proof of the functional determinant on the torus results, which does not use the Kronecker first limit formula nor the functional equation of the non-holomorphic Eisenstein series. As a bonus, several identities involving the Dedekind eta function are obtained as well.

Keywords: Tori; zeta functions; spectral theory; functional determinants.

Mathematics Subject Classification 2010: Primary 11M41, 81T40; Secondary 81T25

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The Chowla-Selberg Formula (CS)


- P. Deligne, *Valeurs de fonctions L et periodes d’integrales*, PSPM 33 (1979) 313-346
ON EPSTEIN'S ZETA FUNCTION (I)

BY S. CHOWLA AND A. SELBERG

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Communicated by H. Weyl, May 18, 1949

1. This paper contains a short account of results whose detailed proofs will be published later.

We define the function $Z(s)$ by

$$Z(s) = \sum'(am^2 + bmn + cn^2)^{-s}$$

(1)

where $s = \sigma + it$ (and $t$, real), $\sigma > 1$, and the summation is for all integers $m$, $n$ (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further $a$ and $c$ are positive numbers while $b$ is real and subject to $4ac - b^2 = \Delta > 0$.

It is well known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole $s$-plane, and satisfies a functional equation similar to the one satisfied by the Riemann Zeta Function. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

Deuring (Math. Ztschr., 37, 403–413 (1933)) obtained an important formula for $Z(s)$. Deuring's work led Heilbronn (Quart. J. Maths., Oxford, 5, 150 (1934)) to the proof of the following famous conjecture of Gauss on the class-number of binary quadratic forms with a negative fundamental discriminant: let $h(-\Delta)$ denote the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, then

$$h(-\Delta) \to \infty \text{ as } \Delta \to \infty$$

(2)

Again using the ideas of Heilbronn and Deuring, Siegel proved that

$$h(-\Delta) > \Delta^{1/2 - \varepsilon} [\Delta > \Delta_0(\varepsilon)]$$

(3)

which is a great advance on (2).

Our starting point is the formula:

$$Z(s) = 2\zeta(2s)a^{-s} + \frac{2^s a^{s-1} \sqrt{\pi}}{\Gamma(s) \Delta^{s-1/2}} 2s(2s - 1) \Gamma(s - 1/2) + Q(s)$$

(4)

where

$$Q(s) = \frac{\pi^s \cdot 2^s + s/2}{a^{1/2} \Gamma(s) \Delta^{s/2 - 1/4}} \sum_{n=1}^{\infty} n^{s - 1/2} \sigma_{1-2s}(n) \cos \left( \frac{n\pi b}{a} \right) \int_{\phi}^{\infty} \phi^{s - 1/2} d\phi$$

(4)
Lerch (1897):

\[
\sum_{\lambda=1}^{\lfloor D \rfloor} \left( \frac{D}{\lambda} \right) \log \Gamma \left( \frac{\lambda}{D} \right) = h \log |D| - \frac{h}{3} \log(2\pi) - \sum_{(a,b,c)} \log a \\
+ \frac{2}{3} \sum_{(a,b,c)} \log [\theta_1'(0|\alpha)\theta_1'(0|\beta)]
\]

$D$ discriminant, $\theta'_1 \sim \eta^3$

$h$ class number of binary quadratic forms $(a, b, c)$

**Eta evaluations** Dedekind eta function for $\text{Im} (\tau) > 0$

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q := e^{2\pi i \tau}
\]

It is a 24-th root of the discriminant func $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \mid a, b \in \mathbb{Z}\}$

\[
\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}
\]

Langlands program: geom of number th
Extended CS Series Formulas (ECS)

Consider the zeta function (\(\text{Re} s > p/2, A > 0, \text{Re} q > 0\))

\[
\zeta_{A, \vec{c}, q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} ' \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{n} \in \mathbb{Z}^p} ' [Q (\vec{n} + \vec{c}) + q]^{-s}
\]

\text{prime: point } \vec{n} = \vec{0} \text{ to be excluded from the sum}

(inescapable condition when } c_1 = \cdots = c_p = q = 0)

\[Q (\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}\]

Case \( q \neq 0 \) (\(\text{Re} q > 0\))

\[
\zeta_{A, \vec{c}, q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s - p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2} \pi^s q^{-s/2+p/4}}{\sqrt{\det A} \Gamma(s)}
\]

\[\times \sum_{\vec{m} \in \mathbb{Z}^{p}_{1/2}} ' \cos(2\pi \vec{m} \cdot \vec{c}) (\vec{m}^T A^{-1} \vec{m})^{s/2-p/4} K_{p/2-s} \left(2\pi \sqrt{2q \vec{m}^T A^{-1} \vec{m}} \right) \]

Pole: \( s = p/2 \) Residue:

\[\text{Res}_{s=p/2} \zeta_{A, \vec{c}, q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}\]
Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane.

Exhibits singularities (simple poles) of the meromorphic continuation— with the corresponding residua— explicitly.

Only condition on matrix $A$: corresponds to (non negative) quadratic form, $Q$. Vector $\vec{c}$ arbitrary, while $q$ is (to start) a non-neg constant.

$K_\nu$ modified Bessel function of the second kind and the subindex $1/2$ in $\mathbb{Z}_{1/2}^p$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$.

[simple criterion: one may select those vectors in $\mathbb{Z}^p \setminus \{0\}$ whose first non-zero component is positive]

Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[ \pi^{j/2} a_{p-j}^{j/2-s} \Gamma \left( s - \frac{j}{2} \right) \zeta_R(2s-j) + \right. $$

$$\left. 4\pi^s a_{p-j}^{s-j/2} \sum_{n=1}^{\infty} \sum_{\vec{m} \in \mathbb{Z}^j} n^{j/2-s} (\vec{m}^t A_j^{-1} \vec{m})^{s/2-j/4} K_{j/2-s} \left( 2\pi n \sqrt{a_{p-j} \vec{m}^t A_j^{-1} \vec{m}} \right) \right]$$

[ECS3d]
Pseudodifferential Operator ($\Psi$ DO)

- **A $\Psi$DO of order** $m$ **$M_n$ manifold**

- **Symbol of $A$:** $a(x, \xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices $\alpha, \beta$ there exists a constant $C_{\alpha, \beta}$ so that

\[
|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|}
\]

**Definition of $A$** (in the distribution sense)

\[
Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x, \xi) \hat{f}(\xi) \, d\xi
\]

- $f$ is a smooth function

\[
f \in S = \{ f \in C^\infty(\mathbb{R}^n); \sup_x |x^\beta \partial^\alpha f(x)| < \infty, \ \forall \alpha, \beta \in \mathbb{N}^n \}
\]

- $S'$ space of tempered distributions

- $\hat{f}$ is the Fourier transform of $f$
The symbol of a $\Psi$DO has the form:

$$a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots + a_{m-j}(x, \xi) + \cdots$$

being $a_k(x, \xi) = b_k(x) \xi^k$

$a(x, \xi)$ is said to be **elliptic** if it is invertible for large $|\xi|$ and if there exists a constant $C$ such that $|a(x, \xi)^{-1}| \leq C(1 + |\xi|)^{-m}$, for $|\xi| \geq C$

An elliptic $\Psi$DO is one with an elliptic symbol

--- $\Psi$DOs are basic tools both in Mathematics & in Physics ---

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]
2. Proof of the Atiyah-Singer index formula
3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]
4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through $\mu$localization (the most important step towards the understanding of linear PDEs since the invention of distributions) [K Fredenhagen, R Brunetti, ... R Wald ’06, R Haag EPJH35 ’10]
Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$
2. $A$ acts on the space of smooth sections of
3. $E$, $n$-dim vector bundle over
4. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:
$$\zeta_A(s) = \text{tr} A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re} s > \frac{n}{m} := s_0$$

\{\lambda_j\} ordered spectrum of $A$, $s_0 = \dim M / \text{ord} A$ abscissa of convergence of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg} \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec} A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$

(d) The only possible singularities of $\zeta_A(s)$ are poles at
$$s_j = (n - j)/m, \quad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$
**Definition of Determinant**

\( H \) \( \Psi DO \) operator \( \{ \varphi_i, \lambda_i \} \) spectral decomposition

\[
\prod_{i \in I} \lambda_i \quad ?! \\
\ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i
\]

Riemann zeta func: \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \text{Re} \ s > 1 \) (& analytic cont)

Definition: zeta function of \( H \)

\( \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s} \)

As Mellin transform: \( \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} \text{tr} \ e^{-tH}, \quad \text{Re} s > s_0 \)

Derivative: \( \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \)

**Determinant:** Ray & Singer, ’67

\[
\det_{\zeta} H = \exp \left[ -\zeta'_H(0) \right]
\]

Weierstrass def: subtract leading behavior of \( \lambda_i \) in \( i \), as \( i \to \infty \), until series \( \sum_{i \in I} \ln \lambda_i \) converges \( \implies \) non-local counterterms !!

C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...
Properties

- The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut.

- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex $s$-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C} \setminus \mathbb{Z}$ (they do not admit spectral cuts), has been obtained \cite{Kontsevich and Vishik}.

- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum_{\lambda \in \text{Spec } A}^\prime e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \text{Res}_{s=s_j} \zeta_A(s), \quad s_j \notin \mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[ \text{PP } \zeta_A(-k) + \psi(k + 1) \text{Res}_{s=-k} \zeta_A(s) \right],$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N} \setminus \{0\}$$

$$s_j = -k, \quad k \in \mathbb{N}$$

$$\text{PP } \phi := \lim_{s \to p} \left[ \phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]$$
“Hi, Emilio. This is a question I have been trying to solve for years. With a bit of luck you could maybe provide me with a hint or two.

• Imagine I’ve got a functional integral and I perform a point transformation (doesn’t involve derivatives). Its Jacobian is a kind of functional determinant, but of a non-elliptic operator (it is simply infinite times multiplication by a function.) Did anybody study this seriously?

• I do know, from at least one paper I did with Luis AG, that in some cases (T duality) one is bound to define something like

\[ \det f(x) \sim \frac{\det [f(x) O]}{\det O} \]

where O es an elliptic operator (e.g. the Laplacian)

• This is what Schwarz and Tseytlin did in order to obtain the dilaton transformation

• And LAG and I did also proceed in a basically similar way

• As I know, Konsevitch, too, uses a related method involving the multiplicative anomaly

Tell me what you know about, please. Thanks so much. - Hugs, Enrique “
Multipl or N-Comm Anomaly, or Defect

Given $A$, $B$, and $AB \psi$ DOs, even if $\zeta_A$, $\zeta_B$, and $\zeta_{AB}$ exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$$
\[ \det_3(AB) = \det_3 A \det_3 B \]

\[ \log \det_3 = \text{tr}_3 \log, \quad \det_3 = e^{\text{tr}_3 \log} \]

\[ \det_3(AB) = e^{\text{tr}_3 \log(AB)} = e^{\text{tr}_3 (\log A + \log B)} \]

\[ = e^{\text{tr}_3 \log A} + e^{\text{tr}_3 \log B} \]

\[ = e^{\text{tr}_3 \log A} \cdot e^{\text{tr}_3 \log B} \]

\[ = \det_3 A \cdot \det_3 B \]

\[[A, B] = 0 \quad \text{assumed!}\]

Which step is wrong?

\[ \text{tr}_3 \text{ is not trace at all} \]

\[ \text{tr}_3 (A, +A_2) \neq \text{tr}_3 A, + \text{tr}_3 A_2 \]

Recall

\[ \text{tr}_3 A = \mathcal{Z}_A(s=-1) = \sum_n x_n^s |_{s=-1} \]
Multipl or N-Comm Anomaly, or Defect

Given $A$, $B$, and $AB$ ψDOs, even if $ζ_A$, $ζ_B$, and $ζ_{AB}$ exist, it turns out that, in general,

$$\det_{ζ}(AB) \neq \det_{ζ}A \cdot \det_{ζ}B$$

The multiplicative (or noncommutative) anomaly (defect) is defined as

$$δ(A, B) = \ln \left[ \frac{\det_{ζ}(AB)}{\det_{ζ}A \cdot \det_{ζ}B} \right] = -ζ'_{AB}(0) + ζ'_{A}(0) + ζ'_{B}(0)$$

Wodzicki formula

$$δ(A, B) = \text{res} \left\{ [\ln \sigma(A, B)]^2 \right\}$$

where

$$σ(A, B) = A^{\text{ord} B} B^{−\text{ord} A}$$
The Dixmier Trace

In order to write down an action in operator language one needs a functional that replaces integration.

For the Yang-Mills theory this is the Dixmier trace.

It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators $T$ such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \quad \mu_0 \geq \mu_1 \geq \cdots$$

Definition of the Dixmier trace of $T$:

$$\text{Dtr } T = \lim_{N \to \infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in $N$ are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$].

The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator $T^{-1}$ at $s = 1$ [Connes]

$$\text{Dtr } T = \lim_{s \to 1^+} (s - 1) \zeta_{T^{-1}}(s)$$
The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to $\Psi$DOs which are not in $L^{(1,\infty)}$.
- Only trace one can define in the algebra of $\Psi$DOs (up to multipl const).
- Definition: $\text{res } A = 2 \text{ Res}_{s=0} \text{ tr}(A \Delta^{-s})$, $\Delta$ Laplacian.
- Satisfies the trace condition: $\text{res } (AB) = \text{res } (BA)$.
- Important!: it can be expressed as an integral (local form):

$$\text{res } A = \int_{S^*M} \text{ tr } a_{-n}(x, \xi) \, d\xi$$

with $S^*M \subset T^*M$ the co-sphere bundle on $M$ (some authors put a coefficient in front of the integral: Adler-Manin residue).

- If $\text{dim } M = n = - \text{ ord } A$ ($M$ compact Riemann, $A$ elliptic, $n \in \mathbb{N}$),
  it coincides with the Dixmier trace, and $\text{Res}_{s=1} \zeta_A(s) = \frac{1}{n} \text{ res } A^{-1}$.

- The Wodzicki residue makes sense for $\Psi$DOs of arbitrary order. Even if the symbols $a_j(x, \xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace.
Consequences of the Multipl Anomaly

In the path integral formulation

\[
\int [d\Phi] \exp \left\{ - \int d^D x \left[ \Phi^\dagger(x)(\cdots)\Phi(x) + \cdots \right] \right\}
\]

Gaussian integration:

\[
\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \longrightarrow \begin{pmatrix} A \\ B \end{pmatrix}
\]

\[
\det(AB) \quad \text{or} \quad \det A \cdot \det B
\]

In a situation where a superselection rule exists, \( AB \) has no sense (much less its determinant):

\[
\Longrightarrow \quad \det A \cdot \det B
\]

But if diagonal form obtained after change of basis (diag. process), the preserved quantity is:

\[
\Longrightarrow \quad \det(AB)
\]
1917

✓ Universe: eternal
✓ Universe: static \textit{why?}
✓ Universe = Milky Way

Einstein field equations with $\Lambda$

\textit{"Eine Größte Eselei"}

Beginning of (Theoretical) Modern Cosmology
\[ F = G \frac{Mm}{r^2} \]

\[ E = mc^2 \]

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \]

\[ \Omega_{\text{tot}} = \Omega_r + \Omega_m + \Omega_k + \Omega_\lambda \]
Trying to solve these puzzles!

- The cc $\lambda$ is indeed a peculiar quantity
- has to do with cosmology Einstein’s eqs., FRW universe
- has to do with the local structure of elementary particle physics stress-energy density $\mu$ of the vacuum

$$L_{cc} = \int d^4x \sqrt{-g} \mu^4 = \frac{1}{8\pi G} \int d^4x \sqrt{-g} \Lambda$$

- In other words: two contributions on the same footing
  [Pauli 20’s, Zel’dovich ’68]

$$\frac{\Lambda c^2}{8\pi G} + \frac{1}{\text{Vol}} \frac{\hbar c}{2} \sum_i \omega_i$$

- For elementary particle physicists: a great embarrassment
  no way to get rid off
  Coleman, Hawking, Weinberg, Polchinski, ... ’88-’89

THE COSMOLOGICAL CONSTANT PROBLEM
Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}$ mm dims, bulk vs brane Susy breaking scales
* T. Padmanabhan, gr-qc/0606061: Holographic Perspective, CC is an intg const, no response of gravity to changes in bulk vac energy dens

We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:

(a) small and large compactified scales
(b) dS & AdS worldbranes
(c) supergraviton theories (discret dims, deconstr)
Uses of zeta regularization in QFT with boundary conditions: a cosmo-topological Casimir effect

Emilio Elizalde

Published 10 May 2006 • 2006 IOP Publishing Ltd

Abstract Zeta regularization has proven to be a powerful and reliable tool for the regularization of the vacuum energy density in ideal situations. With the Hadamard complement, it has been shown to provide finite (and meaningful) answers too in more involved cases, as when imposing physical boundary conditions (BCs) in two- and higher-dimensional surfaces (being able to mimic, in a very convenient way, other ad hoc cut-offs, as non-zero depths). Recently, these techniques have been used in calculations of the contribution of the vacuum energy of the quantum fields pervading the universe to the cosmological constant (cc). Naive counting of the absolute contributions of the known fields lead to a value which is off by as much as 120 orders of magnitude, as compared with observational tests, what is known as the cosmological constant problem. This is very difficult to solve and we do not address that question directly. What we have considered—with relative success in several approaches of different nature—is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models (kind of cosmological Casimir effects). Assuming someone will be able to prove (some day) that the ground value of the cc is zero, as many had suspected until very recently, we will then be left with this incremental value coming from the topology or BCs. We show that this value can have the correct order of magnitude—corresponding to the one coming from the observed acceleration in the expansion of our universe—in a number of quite reasonable models involving small and large compactified scales and/or brane BCs, and supergravitons.

Quantum vacuum fluctuations and the cosmological constant

Emilio Elizalde

Published 6 June 2007 • 2007 IOP Publishing Ltd

Abstract Zeta function regularization techniques are optimally suited for the calculation of the contribution of fluctuations of the vacuum energy of the quantum fields pervading the universe to the cosmological constant (cc). The order of magnitude calculations of the absolute contributions of all fields is known to lead to a value which is off by over 120 orders, as compared with the results obtained from observational fits, known as the new cc problem. This is difficult to solve and many authors still stick to the old problem to try to prove that basically its value is zero with some perturbations thereof leading to the (small) observed result (Burgess C P et al 2006 Preprints hep-th/0606020, 0510123, Padmanabhan T 2006 Preprint gr-qc/0606061, etc). We also address this issue in a somewhat similar way, by considering the additional contributions to the cc that may come from the possibly non-trivial topology of space and from specific boundary conditions imposed on braneworld and other seemingly reasonable models that are being considered in the literature (mainly with other purposes too)—kind of a Casimir effect at cosmological scale. If the ground value of the cc would be indeed zero, we would then be left with this perturbative quantity coming from the topology or BCs. We review the status of this approach, in particular the fact that the computed number is of the right order of magnitude (and has the right sign, what is also non-trivial) when compared with the observational value, in some of the aforementioned examples.
**Zeta function methods and quantum fluctuations**

Emilio Elizalde
Published 15 July 2008 • 2008 IOP Publishing Ltd

*Journal of Physics A: Mathematical and Theoretical, Volume 41, Number 30*

**Abstract**

A review of some recent advances in zeta function techniques is given, in problems of pure mathematical nature but also as applied to the computation of quantum vacuum fluctuations in different field theories, and specially with a view to cosmological applications.

- 385 Total downloads
- Cited by 17 articles

**LETTER TO THE EDITOR**

**On the issue of imposing boundary conditions on quantum fields**

E Elizalde
Published 29 October 2003 • 2003 IOP Publishing Ltd

*Journal of Physics A: Mathematical and General, Volume 36, Number 45*

**Abstract**

An interesting example of the deep interrelation between physics and mathematics is obtained when trying to impose mathematical boundary conditions on physical quantum fields. This procedure has recently been re-examined with care. Comments on that and previous analysis are provided here, together with considerations on the results of the purely mathematical zeta-function method, in an attempt at clarifying the issue. Hadamard regularization is invoked in order to fill the gap between the infinities appearing in the QFT renormalized results and the finite values obtained in the literature with other procedures.

- 318 Total downloads
- Cited by 17 articles

**Dynamical Casimir effect with semi-transparent mirrors, and cosmology**

Emilio Elizalde
Published 9 April 2008 • 2008 IOP Publishing Ltd

*Journal of Physics A: Mathematical and Theoretical, Volume 41, Number 16*

**Abstract**

After reviewing some essential features of the Casimir effect and, specifically, of its regularization by zeta function and Hadamard methods, we consider the dynamical Casimir effect (or Fulling–Davies theory), where related regularization problems appear, with a view to an experimental verification of this theory. We finish with a discussion of the possible contribution of vacuum fluctuations to dark energy, in a Casimir-like fashion, that might involve the dynamical version.

264 Total downloads
2.1. Regularization of the vacuum energy density

For a \((p,q)\)-toroidal universe, with \(p\) the number of large and \(q\) of small dimensions:

\[
\rho_\phi = \frac{1}{a^p b^q} \sum_{n_p, n_q = -\infty}^{\infty} \left( \frac{1}{a^2} \sum_{j=1}^{p} n_j^2 + \frac{1}{b^2} \sum_{k=1}^{q} m_k^2 + M^2 \right)^{(d+1)/2+1},
\]

which corresponds to all large (resp. all small) compactification scales being the same. The squared mass of the field should be divided by \(4\pi^2\mu^2\), but we have renamed it as \(M^2\) to simplify. We also dismiss the mass-dimension factor \(\mu\), easy to recover later.

For a \((p\text{-toroidal}, q\text{-spherical})\)-universe,

\[
\rho_\phi = \frac{1}{a^p b^q} \sum_{n_p = -\infty}^{\infty} \sum_{n_q = -\infty}^{\infty} P_{q-1}(l) \left( \frac{4\pi^2}{a^2} \sum_{j=1}^{p} n_j^2 + \frac{l(l + q)}{b^2} + M^2 \right)^{(d+1)/2+1},
\]


\(P_{q-1}(l)\) being a polynomial in \(l\) of degree \(q - 1\). We assume that \(d = 3 - p\) is the number of non-compactified, large spatial dimensions, and \(\rho_\phi\) needs to be regularized. We use the zeta function [6], taking advantage of our expressions in [7, 8]. No further subtraction or renormalization is needed (the subtraction at infinity is zero, and not even a finite renormalization shows up). Using the mentioned formulas, that generalize the Chowla-Selberg expression to encompass Eqs. (5) and (6), we can provide arbitrarily accurate results (even for different values of the compactification radii [9]).

For the first case, Eq. (5), we obtain

\[
\rho_\phi = -\frac{1}{a^p b^q + 1} \sum_{h=0}^{p} \left( \frac{b}{h} \right)^2 \sum_{n_k, m_k = -\infty}^{\infty} \sqrt{\sum_{k=1}^{q} m_k^2 + M^2} K_1 \left( \frac{2\pi a}{b} \sum_{j=1}^{h} n_j^2 \right) K_1 \left( \frac{2\pi a}{b} \sum_{k=1}^{q} m_k^2 + M^2 \right).
\]

The only presence of the mass-dimension parameter \(\mu\) is as \(M/\mu\) everywhere, and this does not affect the small-\(M\) limit, \(M/\mu \ll b/a\). Inserting back the \(h\) and \(c\) factors, we get

\[
\rho_\phi = -\frac{\hbar c}{2\pi a^p b^q + 1} \left[ 1 + \sum_{h=0}^{p} \left( \frac{b}{h} \right)^2 \alpha \right] + \mathcal{O} \left( \frac{q}{K_1} \right) \left( \frac{2\pi a}{b} \right),
\]

where \(\alpha\) is a computable finite constant, obtained as an explicit geometrical sum in the limit \(M \to 0\). It is remarkable that we do get a well defined limit, independent of \(M\), provided \(M^2\) is small enough.*

2.2. Numerical results

<table>
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<th>(\rho_\phi)</th>
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<th>(p = 1)</th>
<th>(p = 2)</th>
<th>(p = 3)</th>
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Zeta-function Regularization of Holographic Wilson Loops

Jeremías Aguilera-Damía, Alberto Faraggi, Leopoldo A. Pando Zayas, Vimal Rathee and Guillermo A. Silva

Abstract

Using \( \zeta \)-function regularization, we study the one-loop effective action of fundamental strings in \( AdS_5 \times S^5 \) dual to the latitude \( \frac{1}{4} \)-BPS Wilson loop in \( N = 4 \) Super-Yang-Mills theory. To avoid certain ambiguities inherent to string theory on curved backgrounds we subtract the effective action of the holographic \( \frac{1}{2} \)-BPS Wilson loop. We find agreement with the expected field theory result at first order in the small latitude angle expansion but discrepancies at higher order.
Character Integral Representation of Zeta function in $\text{AdS}_{d+1}$:

I. Derivation of the general formula

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ABSTRACT: The zeta function of an arbitrary field in $(d+1)$-dimensional anti-de Sitter (AdS) spacetime is expressed as an integral transform of the corresponding $\text{so}(2,d)$ representation character, thereby extending the results of [1603.05387] for AdS$_4$ and AdS$_5$ to arbitrary dimensions. The integration in the variables associated with the $\text{so}(d)$ part of the character can be recast into a more explicit form using derivatives. The explicit derivative expressions are presented for AdS$_{d+1}$ with $d = 2, 3, 4, 5, 6$. 
Casimir effect with a helix torus boundary condition

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We use the generalized Chowla-Selberg formula to consider the Casimir effect of a scalar field with a helix torus boundary condition in the flat \((D+1)\)-dimensional spacetime. We obtain the exact results of the Casimir energy density and pressure for any \(D\) for both massless and massive scalar fields. The numerical calculation indicates that once the topology of spacetime is fixed, the ratio of the sizes of the helix will be a decisive factor. There is a critical value \(r_{\text{crit}}\) of the ratio \(r\) of the lengths at which the pressure vanishes. The pressure changes from negative to positive as the ratio \(r\) passes through \(r_{\text{crit}}\) increasingly. In the massive case, we find the pressure tends to the result of massless field when the mass approaches zero. Furthermore, there is another critical ratio of the lengths \(r'_{\text{crit}}\) and the pressure is independent of the mass at \(r = r'_{\text{crit}}\) in the \(D = 3\) case.

Keywords: Casimir effect; zeta function; boundary condition; Chowla-Selberg formula.

PACS Nos.: 02.30.Gp; 11.10.-z

1. Introduction

Casimir’s calculation of the force between two neutral, parallel conducting plates originally inspired much theoretical interest as macroscopic manifestation of quantum fluctuation of the field in vacuum. However, the Casimir effect arises not only in the presence of material boundaries, but also in spaces with non-Euclidean topology\(^1\). The simplest example of the Casimir effect of topological origin is the scalar field on a flat manifold with topology of a circle \(S^1\). The topology of \(S^1\) causes the periodicity condition \(\phi(t, 0) = \phi(t, C)\) for a Hermitian scale field \(\phi(t, x)\), where \(C\) is the circumference of \(S^1\), imposed on the wave function which is of the same kind as those due to boundary and resulting in an attractive Casimir force. Similarly, the antiperiodic conditions can be drawn on a Möbius strip and bring about the repulsive Casimir force as a result. Recently, the topology of the helix boundary conditions is investigated in ref\(^2\). We find that the Casimir effect is very

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Casimir Energies of Cylinders: Universal Function

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Abstract

New exact results are given for the interior Casimir energies of infinitely long waveguides of triangular cross section (equilateral, hemiequilateral, and isosceles right triangles). Results for cylinders of rectangular cross section are rederived. In particular, results are obtained for interior modes belonging to Dirichlet and Neumann boundary conditions (TM and TE modes). These results are expressed in rapidly convergent series using the Chowla-Selberg formula, and in fact may be given in closed form, except for general rectangles. The energies are finite because only the first three heat-kernel coefficients can be nonzero for the case of polygonal boundaries. What appears to be a universal behavior of the Casimir energy as a function of the shape of the regular or quasi-regular cross-sectional figure is presented. Furthermore, numerical calculations for arbitrary right triangular cross sections suggest that the universal behavior may be extended to waveguides of general polygonal cross sections. The new exact and numerical results are compared with the proximity force approximation (PFA).

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Casimir effect in Yang-Mills theory

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We study, for the first time, the Casimir effect in non-Abelian gauge theory using first-principle numerical simulations. Working in two spatial dimensions at zero temperature we find that closely spaced perfect chromoelectric conductors attract each other with a small anomalous scaling dimension. At large separation between the conductors, the attraction is exponentially suppressed by a new massive quantity, the Casimir mass, which is surprisingly different from the lowest glueball mass. The apparent emergence of the new massive scale may be a result of the backreaction of the vacuum to the presence of the plates as sufficiently close chromoelectric conductors induce, in a space between them, a smooth crossover transition to a color deconfinement phase.

Quantum fluctuations of virtual particles are affected by the presence of physical objects. This property is a cornerstone of the Casimir effect [1] which states that the energy of vacuum ("zero-point") quantum fluctuations should be modified by the presence of physical bodies [2, 3]. The energy shift of the virtual particles has real physical consequences because the Casimir effect leads to non-trivial changes of fundamental properties of the system depend on the geometry of the system. The interactions may even change the overall sign of the Casimir–Polder force in certain fermionic systems with condensates [18] and in the $\mathbb{CP}^{N-1}$ model on an interval [19, 20]. First-principle numerical simulations show that the presence of the boundaries affects also non-perturbative (de)confining properties of certain bosonic systems [19, 21].
You found that a finite-geometry may lead to a certain phase transition

Actually, *what kind of transition should it be?*

- Savvidy vacuum implies quark confinement via randomized color-magnetic fluxes
- In our case, a classical minimum for the magnetic flux correlated with compactified directions
- A partial (de)confinement, corresponding to compactified axes of space

Thank you! With best regards,

Maxim Chernodub, Vladimir Goy, Alexander Molochkov, Ha Nguyen

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Analytic Aspects of Quantum Fields

A. A. Bytsenko
G. Cognola
E. Elizalde
V. Moretti
S. Zerbini

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The meaning of Big Bang

There are too many misconceptions about the original meaning of the expression Big Bang. Going back to the origins of the origin of everything, it turns out that, contrary to what we find in so many places, Fred Hoyle was not the very first to pronounce these two words together—in a cosmological context— in his famous BBC lecture of the year 1949. Actually, since the discovery during the late 1920s of the fact that distant spiral nebulae were speeding away from us at very high velocities—proportional to their distances (Hubble’s law)—many cosmologists got convinced that, at some point back in the past, an explosion of some kind need have necessarily occurred. This would be responsible for having set out all those celestial objects in recession.

Cambridge astronomers and theoreticians, and this included the famous Arthur Eddington, often used the expressions Bang and Big Bang during the 1930s, in order to designate that cosmic explosion. To repeat, these terms had, therefore, the meaning of an original thrust, or big thrust, produced by some kind of cosmic explosion or instantaneous force of unknown nature, which was necessary in order to explain the high recession speeds of the galaxies. Vesto Slipher had reported this recession in 1914 already, in the year’s meeting of the American Astronomical Society. Definitely, he was the first to discover that the beautiful model of the universe, eternal and static, was in serious trouble. This was a very remarkable observation and he was so convincing that his talk received from the audience, chronicles say, a long and standing applause.

It is thus clear that Hoyle did not invent the term Big Bang, in his acclaimed popular talk on the BBC of March 28th, 1949. However, he did give to it a radically different meaning, with scientific roots embedded in the deepest principles of Einstein’s Theory of General Relativity. Even today, after 70 years, only specialists in this theory can actually understand the meaning of his extremely precise words. Thus spoke Hoyle:

[Lemaître’s model implies that] “... all matter in the universe was created in one Big Bang at a particular time ...”

He pronounced these words, reports say, with an intonation clearly meaning that this fact was completely impossible, utterly absurd.

Fred Hoyle had been the first person to discover that all of us are stardust. This is, that most of the elements, of the atoms in our body could not have been formed in this initial stage of the cosmos, but only much later in its evolution, after galaxies appeared and stars evolved, in explosions of novae and supernovae (the now standard theory of stellar nucleosynthesis, which he pioneered) [Very recent work sets the formation of the first stars at about 180 million years and the first supernova explosions at about 80 million years later, in a colder than expected medium which seems to point to the presence of dark matter.] In his obituary “Stardust memories”, written by John Gribbin and published in The Independent in 2005, there is a nice account of all that.

With Gold and Bondi, Hoyle had proposed the Steady State Theory, in an attempt at recomposing the static model of the universe that had reigned without rival until the already mentioned discoveries. To keep the cosmic energy density constant, in spite of the recession of the galaxies, matter and energy had to be created in their model, what was done by means of a creation field, out of nothing. This happened in faraway regions of the cosmos, through expansions of the fabric of space, in small proportions (e.g., many little Bangs); just in order to accurately compensate the decrease of the matter/energy density. Such creation seemed perfectly reasonable using GR. What was completely crazy (for Hoyle) was to imagine that “all matter in the universe” could have been created in only “one Big Bang” in a short instant of time in the past. Only a crazy mind would imagine this possibility. Such is the precise meaning of Hoyle’s sentence, word by word (his particular intonation included).

By now it should be clear to the reader that Hoyle gave to the term Big Bang a completely different meaning from which it had had in Cambridge until that date. Indeed, from being an ordinary explosion, which simply set in motion the pre-existing masses of the cosmos, he converted it into a creation push, and incredibly huge expansion of the fabric of space, an enormous instantaneous negative pressure, which would allow for the possibility of the creation of the formidable amount of positive mass and energy of the whole universe. And all this starting out of nothing, in a unique creation blow, one Big Bang. In fact, Einstein’s theory allows for this to happen (and many other things) without breaking at any stage the energy conservation principle (energy balance). However, the big question was, what precise mechanism could be invoked as responsible for such enormous blow up? No one, according to Hoyle.
But, alas, exactly thirty years later, an American PhD in theoretical physics appeared, with name Allan Guth, who was about to finish his last Post-Doc contract, and thus on the verge of being expelled from the American University system. Faced up with the imperious necessity to make some extraordinary discovery, and quickly, he managed to give birth to a brand new revolutionary theory, which he named Cosmic Inflation. With it, Guth was not just able to do what Hoyle considered as absolutely impossible, but, on top of it and in a single stroke, he solved all the endemic problems of the universe models with an origin, which had been accumulating during the preceding decades (like the horizon, causality, and absence of monopoles problems). The Physics upon which Guth grounded his theory was exactly the same that Hoyle and his colleagues had invoked in their Steady State Theory, namely (the reader will have surely guessed it by now) no other than General Relativity.

Until his death, Hoyle strongly defended this similarity, proclaiming on many occasions that inflation had not much more in it than his old theory of many years ago. This is not true, by any means. What is indeed true is the fact that the deep roots, the fundamental principles on which both theories stand are exactly the same. However, inflationary models, which we now count by the dozens, are much more elaborated and predictive.

Anyhow, the fact that there are so many models of inflation is not actually desirable. Allow me a last reflection. When we compare this with the situation of one hundred years ago, with the extraordinary beauty of Einstein’s theory, we feel a bit disappointed. When he formulated it in 1915, General Relativity (as Special Relativity, before) was the result of pure human logic, with just the help of his ‘most happy thought’, namely the equivalence principle, and of the observational fact that the speed of light was constant. With the aid of a couple of extra (mathematical) considerations, the theory was unique, that is, the only possible theory for the universe, but for the simple addition (or subtraction) of a pure constant, the now so famous cosmological constant. He actually introduced it in 1917 in order to cope with the existing static model of the cosmos, discussed above.

Now, in comparison, the situation is much darker (to say it in modern terms). Even more, if we take into account that, additionally, Einstein’s field equations have just one family of solutions describing a universe like ours, homogeneous and isotropic at large scale: the very famous Friedmann-Lemaître-Robertson-Walker model with cold dark matter and cosmological constant (this accounts for the dark energy component, which, as with dark matter, I have not discussed here). It is what we call nowadays the Standard Cosmological Model.

It is this model that has to be supplemented with a theory of inflation. Now, the question is: with which one?

Emilio Elizalde

For more details: arXiv:1801.09550 [physics.hist-ph]

For Misao Sasaki,
on his retiring from YITP,
with very best wishes
for a long and happy life
Thank You

Muitas Grazias