Quantum Vacuum Fluctuations at the Cosmological Level

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“Dynamics and Thermodynamics of Black Holes and Naked Singularities”

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Outline of (an ideal) presentation

- \( \Psi \)DOs, Zeta Functions, Determinants, and Traces
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- ΨDOs, Zeta Functions, Determinants, and Traces
- Wodzicki Residue, Multiplicative (or Noncommutative) Anomaly, or Defect
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- The Chowla-Selberg Expansion Formula (CS) & Extended Expressions (ECS)
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Physics:
- Quantum vacuum fluctuations
  - Laboratory (Dynamical CE) J Haro, EE, PRL’06
  - Actuators (Technology) F Capasso et al, R Onofrio
- At Cosmological Scale
- Non-commutative QFTs (quadratic standard case)
- New developments (quadratic non-standard cases)
A $\Psi$DO of order $m$ $M_n$ manifold

Symbol of $A$: $a(x, \xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices $\alpha, \beta$ there exists a constant $C_{\alpha, \beta}$ so that

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|}$$
Pseudodifferential Operator (ΨDO)

- **A ΨDO of order** $m$ **on** $M_n$ **manifold**

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\[
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\]

**Definition of** $A$ **(in the distribution sense)**

\[
Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x, \xi) \hat{f}(\xi) \, d\xi
\]

- $f$ **is a smooth function**

\[
f \in S = \{ f \in C^\infty(\mathbb{R}^n); \sup_x |x^\beta \partial^\alpha f(x)| < \infty, \forall \alpha, \beta \in \mathbb{N}^n \}
\]

- $S'$ **space of tempered distributions**

- $\hat{f}$ **is the Fourier transform of** $f$
ψDOs are useful tools

The symbol of a ψDO has the form:

\[ a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots + a_{m-j}(x, \xi) + \cdots \]

being \( a_k(x, \xi) = b_k(x) \xi^k \)

\( a(x, \xi) \) is said to be elliptic if it is invertible for large \(|\xi|\) and if there exists a constant \( C \) such that \(|a(x, \xi)^{-1}| \leq C(1 + |\xi|)^{-m} \), for \(|\xi| \geq C\)

- An elliptic ψDO is one with an elliptic symbol
\textbf{\(\Psi\)DOs are useful tools}

The symbol of a \(\Psi\)DO has the form:

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- An elliptic \(\Psi\)DO is one with an elliptic symbol

--- \(\Psi\)DOs are basic tools both in Mathematics & in Physics ---

1. Proof of \textbf{uniqueness of Cauchy problem} \ [Calderón-Zygmund]

2. Proof of the \textbf{Atiyah-Singer index formula}

3. In QFT they appear in any analytical continuation process —as \textbf{complex powers of differential operators}, like the Laplacian \ [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through \(\mu\)\textbf{localization} (the most important step towards the understanding of linear PDEs since the invention of distributions) \ [Fredenhagen, Brunetti, ... R. Wald '06]
Existence of $\zeta_A$ for $A$ a $\Psi DO$

1. $A$ a positive-definite elliptic $\Psi DO$ of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of

3. $E$, $n$-dim vector bundle over

4. $M$ closed $n$-dim manifold
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(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_{j} \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0$$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \ \text{Arg} \ \lambda = \theta, \ \theta_1 < \theta < \theta_2\}$, $\text{Spec} A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)
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(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$

(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \quad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$
DOs on boundaryless manifolds

$M$ compact $n$-dim $C^\infty$ manifold without a boundary, provided with a smooth volume element
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- \( e^{-tA} \) solution operator \( e^{-tA} : f \mapsto u \) for the heat equation
  \[ \partial_t u + A u = 0 \]
  with initial value \( u|_{t=0} = f \)
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- $e^{-tA}$ solution operator $e^{-tA} : f \mapsto u$ for the heat equation $\partial_t u + Au = 0$ with initial value $u|_{t=0} = f$

- This operator is traceclass $\forall t > 0$, and as $t \downarrow 0$

$$\text{tr} e^{-tA} \sim \sum_{j=0}^\infty \alpha_j(A)t^{(j-n)/m} + \sum_{k=1}^\infty \beta_k(A)t^k \log t$$
\[ \Psi \text{DOs on boundaryless manifolds} \]

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- By Mellin transform:
  \[ \zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tA} t^{s-1} \, dt \]
\( \zeta_A(s) \) has a meromorphic extension with only possible poles at

\[
s_j = (n - j)/m, \ j \in \mathbb{N},
\]

at most simple at \( s_j \notin -\mathbb{N} \), at most double at \( s_j \in -\mathbb{N} \)
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Moreover

\[ \alpha_j(A) = \text{Res}_{s=s_j} \Gamma(s) \zeta_A(s), \quad \beta_k(A) = \text{Res}_{s=-k(s+k)} \Gamma(s) \zeta_A(s) \]
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If \( A \) is a diff operator, then: \( \alpha_j(A) = 0, \ j \text{ odd}, \ \beta_k(A) = 0, \forall k \)
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If \( A \geq 0 \) all holds for \( A - \text{Ker}A \), subtract DimKer to Res at 0
\[ \zeta_A(s) \text{ has a meromorphic extension with only possible poles at } \]
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If \( s_j \in \mathbb{N} \), then \( \alpha_j(A) \) is not locally computable

G. Cognola, L. Vanzo, S. Zerbini, JMP, 1992
Definition of Determinant

$H \Psi DO$ operator $\{\varphi_i, \lambda_i\}$ spectral decomposition
Definition of Determinant

\[ H \Rightarrow \Psi \text{DO operator} \quad \{\varphi_i, \lambda_i\} \quad \text{spectral decomposition} \]

\[ \prod_{i \in I} \lambda_i \quad ?! \quad \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]
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Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \ Re \ s > 1$ (& analytic cont)

Definition: zeta function of $H$ $\zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} \ \text{tr} \ e^{-tH}, \ Re \ s > s_0$

Derivative: $\zeta'_H(0) = - \sum_{i \in I} \ln \lambda_i$
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Derivative: \[ \zeta'_H(0) = -\sum_{i \in I} \ln \lambda_i \]

Determinant: \[ \text{Ray & Singer, '67} \quad \det_\zeta H = \exp \left[ -\zeta'_H(0) \right] \]
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Weierstrass def: subtract leading behavior of \( \lambda_i \) in \( i \), as \( i \to \infty \), until series \( \sum_{i \in I} \ln \lambda_i \) converges \( \implies \) non-local counterterms !!
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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...
Properties

The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut.
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Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \text{Res}_{s=s_j} \zeta_A(s), \quad s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[ \text{PP } \zeta_A(-k) + \psi(k + 1) \text{Res}_{s=-k} \zeta_A(s) \right],$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N} \setminus \{0\} \quad s_j = -k, \quad k \in \mathbb{N}$$

$$\text{PP } \phi := \lim_{s \rightarrow p} \left[ \phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]$$
In order to write down an action in operator language one needs a functional that replaces integration.
The Dixmier Trace

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For the Yang-Mills theory this is the **Dixmier trace**

It is the **unique** extension of the usual trace to the ideal $L^{(1, \infty)}$ of the compact operators $T$ such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = O(\log N), \quad \mu_0 \geq \mu_1 \geq \cdots$$
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Definition of the Dixmier trace of $T$:

$$\text{Dtr} \ T = \lim_{N \to \infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in $N$ are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]
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The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator \( T^{-1} \) at \( s = 1 \) [Connes]:

\[
\text{Dtr} \ T = \lim_{s \to 1^+} (s - 1) \zeta_{T^{-1}}(s)
\]
The Wodzicki Residue

The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to \( \Psi \)DOs which are not in \( \mathcal{L}^{(1,\infty)} \).
The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to $\Psi$DOs which are not in $\mathcal{L}^{(1,\infty)}$.

- Only trace one can define in the algebra of $\Psi$DOs (up to multipl const).
The Wodzicki Residue

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Important!: it can be expressed as an integral (local form)

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- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x, \xi), j < m$, are not coordinate invariant, the integral is, and defines a trace
Singularities of $\zeta_A$

A complete determination of the meromorphic structure of some zeta functions in the complex plane can be also obtained by means of the Dixmier trace and the Wodzicki residue.
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- Proposition. Under the conditions of existence of the zeta function of $A$, given above, and being the symbol $a(x, \xi)$ of the operator $A$ analytic in $\xi^{-1}$ at $\xi^{-1} = 0$:

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**Proof.** The homog component of degree $-n$ of the corresp power of the principal symbol of $A$ is obtained by the appropriate derivative of a power of the symbol with respect to $\xi^{-1}$ at $\xi^{-1} = 0$:

$$a_{-n}^{-s_k} (x, \xi) = \left( \frac{\partial}{\partial \xi^{-1}} \right)^k \left[ \xi^{n-k} a^{(k-n)/m} (x, \xi) \right] \bigg|_{\xi^{-1}=0}^{\xi^{-n}}$$
Multipl or N-Comm Anomaly, or Defect

Given $A$, $B$, and $AB$ ψ DOs, even if $ζ_A$, $ζ_B$, and $ζ_{AB}$ exist, it turns out that, in general,

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The multiplicative (or noncommutative) anomaly (defect) is defined as

$$δ(A, B) = \ln \left[ \frac{\text{det}_ζ(AB)}{\text{det}_ζ A \text{det}_ζ B} \right] = -ζ'_{AB}(0) + ζ'_A(0) + ζ'_B(0)$$
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$$δ(A, B) = \text{res} \left\{ \left[ \ln σ(A, B) \right]^2 \right\} \over 2 \text{ord } A \text{ ord } B \left( \text{ord } A + \text{ord } B \right)$$

where

$$σ(A, B) = A^{\text{ord } B} B^{\text{ord } A}$$
In the path integral formulation

\[ \int [d\Phi] \exp \left\{ - \int d^Dx \left[ \Phi^\dagger(x) \Phi(x) + \cdots \right] \right\} \]

Gaussian integration: \[ \rightarrow \quad \det (\ )^{\pm} \]

\[
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A_1 & A_2 \\
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\end{pmatrix}
\rightarrow
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But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: \( \Rightarrow \) \( \det(AB) \)
The Chowla-Selberg Formula (CS)

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- B.H. Gross, On the periods of abelian integrals and a formula of Chowla and Selberg, Inv. Math. 45 (1978) 193-211
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- P. Deligne, Valeurs de fonctions L et periodes d’integrales, PSPM 33 (1979) 313-346
Lerch (1897):

\[ \sum_{\lambda=1}^{[D]} \left( \frac{D}{\lambda} \right) \log \Gamma \left( \frac{\lambda}{D} \right) = h \log |D| - \frac{h}{3} \log (2\pi) - \sum_{(a,b,c)} \log a \]

\[ + \frac{2}{3} \sum_{(a,b,c)} \log \left[ \theta_1'(0|\alpha)\theta_1'(0|\beta) \right] \]

\( D \) discriminant, \( \theta_1' \sim \eta^3 \)

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**Eta evaluations**  
Dedekind eta function for \( \text{Im} (\tau) > 0 \)

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q := e^{2\pi i \tau}
\]

It is a 24-th root of the discriminant function \( \Delta(\tau) \) of an elliptic curve \( \mathbb{C}/L \) from a lattice \( L = \{a\tau + b \mid a, b \in \mathbb{Z}\} \)

\[
\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}
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The C-S formula gives the value of a product of eta functions.
Properties & Recent Results

⇒ The C-S formula gives the value of a product of eta functions

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Basic strategies

- **Jacobi’s identity** for the \( \theta \)-function

\[
\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2n\tau z), \quad q := e^{i\pi \tau}, \ \tau \in \mathbb{C}
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equivalently:

\[
\sum_{n=-\infty}^{\infty} e^{-(n+z)^2t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \cos(2\pi n z), \quad z, t \in \mathbb{C}, |\text{Re } t| > 0
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- Higher dimensions: **Poisson summ formula** (Riemann)

\[
\sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \tilde{f}(\vec{m})
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$\tilde{f}$ Fourier transform

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- **Truncated sums** $\rightarrow$ asymptotic series
Consider the zeta function (Re $s > p/2$, $A > 0$, Re $q > 0$)

\[
\zeta_{A, \vec{c}, q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} = \sum_{\vec{n} \in \mathbb{Z}^p} \left[ Q (\vec{n} + \vec{c}) + q \right]^{-s}
\]

prime: point $\vec{n} = \vec{0}$ to be excluded from the sum

(inescapable condition when $c_1 = \cdots = c_p = q = 0$)

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**Case** \(q \neq 0 (\Re q > 0)\)

\[
\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}q^{p/2-s}}{\sqrt{\det A}} \Gamma(s - p/2) \frac{\Gamma(s)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A} \Gamma(s)}
\]

\[
\times \sum'_{\vec{m} \in \mathbb{Z}_1^{p}} \cos(2\pi \vec{m} \cdot \vec{c}) \left( \vec{m}^T A^{-1} \vec{m} \right)^{s/2-p/4} K_{p/2-s} \left( 2\pi \sqrt{2q \vec{m}^T A^{-1} \vec{m}} \right)
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[EC1]
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**Pole:** \(s = p/2\) \hspace{1cm} **Residue:**

\[
\text{Res}_{s=p/2} \zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\text{det } A)^{-1/2}
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$K_\nu$ modified Bessel function of the second kind and the subindex $1/2$ in $\mathbb{Z}_{1/2}^p$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$.

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Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

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\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[ \pi^{j/2} a_{p-j}^{j/2-s} \Gamma \left( s - \frac{j}{2} \right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{-\frac{j}{2}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j} n^{j/2-s} \left( \vec{m}_j^t A_j^{-1} \vec{m}_j \right)^{s/2-j/4} K_{j/2-s} \left( 2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j} \right) \right]
\]
QFT in s-t with non-comm toroidal part

$D$–dim non-commut manifold: $M = \mathbb{R}^{1,d} \otimes T^p_\theta$, $D = d + p + 1$

$T^p_\theta$ a $p$–dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$

$\sigma_{jk}$ a real, nonsingular, antisymmetric matrix of $\pm 1$ entries

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- Interest recently, in connection with \( M \)-theory & string theory
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- Interest recently, in connection with $M$–theory & string theory

[Connes, Douglas, Seiberg, Cheung, Chu, Chomerus, Ardalan, …]

- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s) = \frac{V \Gamma(s - (d + 1)/2)}{(4\pi)^{(d+1)/2} \Gamma(s)} \sum_{\vec{n}\in\mathbb{Z}^p} \left[1 + \Lambda \theta^2 - 2\alpha Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s}$$

$\alpha = 2$ bos, $\alpha = 3$ ferm, $V = Vol(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j = a_j^{-1/2}$ compactific radii
After some calculations,

\[ \zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - \frac{d+1}{2})}{l! \Gamma(s)} \left(-\Lambda \theta^{2-2\alpha}\right)^l \zeta_{Q,\vec{0},0}(s + \alpha l - \frac{d + 1}{2}) \]

for all radii equal to \( R \), with \( I(\vec{n}) = \sum_{j=1}^{p} n_j^2 \),

\[ \zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2} R^{d+1-2s}} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - \frac{d+1}{2})}{l! \Gamma(s)} \left(-\Lambda \theta^{2-2\alpha}\right)^l \zeta_{E}(s + \alpha l - \frac{d + 1}{2}) \]

where we use the notation \( \zeta_{E}(s) := \zeta_{I,\vec{0},0}(s) \)

e.g., the Epstein zeta function for the standard quadratic form
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**Rich pole structure:** pole of Epstein zf at

\[ s = \frac{p}{2} - \alpha k + \frac{(d+1)}{2} = \frac{D}{2} - \alpha k \]

combined with poles of \( \Gamma \), yields a rich pattern of singul for \( \zeta_{\alpha}(s) \)
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combined with poles of \( \Gamma \), yields a rich pattern of singul for \( \zeta_\alpha(s) \)

**Classify** the different possible cases according to the values of \( d \) and \( D = d + p + 1 \). We obtain, at \( s = 0 \):
For $d = 2k$
\[
\begin{cases}
  \text{if } D \neq \frac{\dot{2}}{2} \alpha & \Rightarrow \zeta_{\alpha}(0) = 0 \\
  \text{if } D = \frac{\dot{2}}{2} \alpha & \Rightarrow \zeta_{\alpha}(0) = \text{finite}
\end{cases}
\]

For $d = 2k - 1$
\[
\begin{cases}
  \text{if } D \neq \frac{\dot{2}}{2} \alpha & \begin{cases}
    \text{finite, for } l \leq k \\
    0, \quad \text{for } l > k
  \end{cases} \Rightarrow \zeta_{\alpha}(0) = \text{finite} \\
  \text{if } D = 2\alpha l & \begin{cases}
    \text{pole, for } l \leq k \\
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\end{cases}
\]

- Pole structure of the zeta function $\zeta_{\alpha}(s)$, at $s = 0$, according to the different possible values of $d$ and $D$ ($\frac{\dot{2}}{2} \alpha$ means multiple of $2\alpha$)
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$\Rightarrow$ Explicit analytic continuation of $\zeta_{\alpha}(s)$, $\alpha = 2, 3$, & specific pole structure
\[ \zeta_\alpha(s) = \frac{2^{s-d} V}{(2\pi)^{(d+1)/2} \Gamma(s)} \sum_{l=0}^{\infty} \frac{\Gamma(s + l - (d + 1)/2)}{l! \Gamma(s + \alpha l - (d + 1)/2)} (-2^\alpha \Lambda^2 \theta^{2-2\alpha})^l \sum_{j=0}^{p-1} (\det A_j)^{-\frac{1}{2}} \]

\times \left[ \frac{\pi^{j/2} a_{p-j}}{a_{p-j}} \right] \Gamma(s + \alpha l - (d + j + 1)/2) \zeta_R(2s + 2\alpha l - d - j - 1)

+ 4\pi^{s+\alpha l-(d+1)/2} a_{p-j}^{-\alpha l/2-(d+j+1)/4} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j} ' n_{(d+j+1)/2-s-\alpha l}

\times \left( \vec{m}_j A_j^{-1} \vec{m}_j \right)^{s+\alpha l/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l} \left( 2\pi n \sqrt{a_{p-j} \vec{m}_j A_j^{-1} \vec{m}_j} \right) \]
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\[
\times \left[ \pi^{j/2} a_{p-j}^{-(s-\alpha l + (d+j+1)/2)} \Gamma(s + \alpha l - (d + j + 1)/2) \zeta_R(2s + 2\alpha l - d - j - 1) \right. \\
+ 4\pi^{s+\alpha l -(d+1)/2} a_{p-j}^{-(s+\alpha l)/2 -(d+j+1)/4} \sum_{n=1}^{\infty} \sum_{\bar{m}_j \in \mathbb{Z}^j} \tilde{\nu}^{(d+j+1)/2-s-\alpha l}
\left. \times \left( \bar{m}_j^t A_j^{-1} \bar{m}_j \right)^{(s+\alpha l)/2 -(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l} \right]

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<table>
<thead>
<tr>
<th>p \ D</th>
<th>even</th>
<th>odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>(1a) pole / finite ((l \geq l_1))</td>
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</tr>
<tr>
<td>even</td>
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- **General pole structure** of \( \zeta_\alpha(s) \), for the possible values of \( D \) and \( p \) being odd or even. **Magenta**, type of behavior corresponding to lower values of \( l \); behavior in **blue** corresponds to larger values of \( l \).
Quantum Vacuum Fluct’s & the CC

The main issue:

energy ALWAYS gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor

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Appears on the rhs of Einstein’s equations:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G (\tilde{T}_{\mu\nu} - \mathcal{E} g_{\mu\nu}) \]

It affects cosmology: \( \tilde{T}_{\mu\nu} \) excitations above the vacuum
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Idea: zero point fluctuations can contribute to the cosmological constant

Ya.B. Zeldovich ’68
Relativistic field: collection of harmonic oscill’s (scalar field)

\[ E_0 = \frac{\hbar c}{2} \sum_{n} \omega_n, \quad \omega = k^2 + \frac{m^2}{\hbar^2}, \quad k = \frac{2\pi}{\lambda} \]
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Evaluating in a box and putting a cut-off at maximum \( k_{\text{max}} \) corresp’ng to QFT physics (e.g., Planck energy)

\[ \rho \sim \frac{\hbar k_{\text{Planck}}^4}{16\pi^2} \sim 10^{123} \rho_{\text{obs}} \]

kind of a modern (and thick!) aether  

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What we do consider —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:

\[ \implies \text{kind of cosmological Casimir effect} \]
Cosmo-Topological Casimir Effect

Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

* L. Parker & A. Raval, VCDM, vacuum energy density
* C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two $10^{-2}$mm dims, bulk vs brane Susy breaking scales
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- (c) supergraviton theories (discret dims, deconstr)
A. Simple model: large & small dim’s

Space-time: $\mathbb{R}^{d+1} \times T^p \times T^q$, $\mathbb{R}^{d+1} \times T^p \times S^q$, ...
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- Scalar field, \( \phi \), pervading the universe (\( \hbar = c = 1 \))

\[
S = \frac{1}{2} \int d^4x \sqrt{-\text{g}} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* + (m^2 + \xi R) \phi \phi^* \right]
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- **$M$** effective mass term, $m$ arbitrarily small

(a tiny mass for the field cannot be excluded, and fits well)

* A. Chodos & E. Myers, ’85–’86
Some recent developments

Braneworld models based on a 6D SUGRA [Gibbons, 2003]
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- We produce an exact analysis of the one-loop effective action in the 4D, alternative model [EE-Minamitsuji-Naylor, PRD 2007]
The one-loop effective potential for the volume modulus is similar to the Coleman-Weinberg potential

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- \( A_4 \) and \( B_4 \) are functions of the model parameters and the shape modulus (brane tensions)

- Stability determined by sign of \( B_4 \): heat-kernel coeff

- Phenomenological effects on the brane (eff mass of modulus, degree of hierarchy between fundamental energy scales on brane): need to know the value of \( A_4 \)
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- Accurate evaluation of \( A_4 \) is crucial for making physical predictions: hierarchy & CC problems
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In previous work WKB approx was used. Now exact analysis of the mass spectrum for Kaluza-Klein-like modes (not standard KK modes, because of conical singularities at poles of the two-sphere on the internal dims): rugby-ball frame
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\]

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a = 4, \quad b = \frac{4(1 + r)}{\kappa}, \quad c = \frac{4r}{\kappa^2}, \quad q = -1, \quad \alpha = 1/2
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perform the \( k \)-integr after interchanging the order of integr

$$= \int d^2 x \frac{2\pi g^2 (1-s)}{s-1} \sum_{m,n} [a(m + \alpha)^2 + b(m + \alpha)|n| + cn^2 + q]^{1-s}$$
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\]

where we redefine the terms:

\[
\hat{\alpha}(n) = \alpha + \frac{bn}{2a} = \frac{1}{2} + \frac{(1 + r)|n|}{2\kappa}, \quad \hat{q}(n) = -\frac{n^2}{\kappa^2}(1 - r)^2 - 1
\]
Extended binomial expansion

\[ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [a(m + \alpha)^2 + b(m + \alpha)n + cn^2 + q]^{-s+1} \]

\[ = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [a(m + \hat{\alpha})^2 + \hat{q}]^{-s+1} \]

\[ = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{\Gamma(j + s - 1)}{\Gamma(s - 1) j!} [a(m + \hat{\alpha})^2]^{1-s-j} \hat{q}^j \]
Extended binomial expansion

\[ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[ a(m + \alpha)^2 + b(m + \alpha)n + cn^2 + q \right]^{-s+1} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[ a(m + \hat{\alpha})^2 + \hat{\hat{q}} \right]^{-s+1} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{\Gamma(j + s - 1)}{\Gamma(s - 1) j!} \left[ a(m + \hat{\alpha})^2 \right]^{1-s-j} \hat{\hat{q}}^j \]

\[ = \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} a^{1-s-j} \frac{\Gamma(j + s - 1)}{\Gamma(s - 1) j!} \hat{\hat{q}}^j \zeta_H(2s+2j-2, \hat{\alpha}), \quad \left| \frac{\hat{q}}{a(m + \alpha)^2} \right| < 1 \]
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\[
\zeta(s)\bigg|_{n \neq 0} = \frac{g^2(1-s)}{\pi} \int d^2 x \sum_{j=0}^{\infty} G(j, s) \sum_{n=1}^{\infty} \left[ \frac{n^2}{\kappa^2} (1-r)^2 + 1 \right]^j \zeta_H \left( 2s+2j-2, \frac{1}{2} + \frac{1 + r}{2\kappa} n \right)
\]

\[
G(j, s) \equiv \frac{2^{-2(j+s-1)} \Gamma(s + j - 1)}{\Gamma(s) j!}
\]
Analytic continuation of the $\zeta$ function

\[
P(s) = \frac{g^{2(1-s)}}{\pi} \int d^2 x \sum_{j=0}^{\infty} G(j, s) \sum_{n=1}^{\infty} \left[ \left( \frac{n^2}{\kappa^2} (1 - r)^2 + 1 \right)^j \right.
\]

\[
\times \zeta_H \left( 2s + 2j - 2, \frac{1}{2} + \frac{1 + r}{2\kappa} n \right) - F(n, j; s) \]
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$$\zeta(s) \bigg|_{n \neq 0} = P(s) + \sum_{j=0}^{\infty} G(j, s) \Delta(j, s), \quad \text{analogously} \quad \zeta_0(s)$$
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\Delta(j, s) = \frac{2^{-6+2j+2s} g^{2(1-s)}}{45(2j + 2s - 3) \pi} \int d^2x \kappa^{1+2s} \frac{(1 - r)^{2j}}{(1 + r)^{1+2j+2s}}
\]
\[
\times \left[ \frac{w_0}{\kappa^4} \zeta_R(2s-3) + \frac{w_2(j, s)}{\kappa^2} \zeta_R(2s-1) + w_4(j, s) \zeta_R(2s+1) \right]
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$$\times \left[ \frac{w_0}{\kappa^4} \zeta_R(2s - 3) + \frac{w_2(j, s)}{\kappa^2} \zeta_R(2s - 1) + w_4(j, s) \zeta_R(2s + 1) \right]$$

⇒ Extend our analysis to the cases of

1. 6 dimensions
2. a bulk scalar field with self-interactions and other fields

in the multiplets appearing in the supergravity model