Quantum Vacuum & Cosmology
with a Background of Zeta Functions

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III Russian-Spanish Congress, Santiago de Compostela, 8 -11 Sep 2015
• 1917 Einstein found from his equations of GR a static model of the Univ, by introducing the cc $\Lambda$

• 1917, few months after, de Sitter found a solution which might explain Slipher’s nebular redshifts

• 1922 Friedmann found E’s eqs allowed a dynamic Univ, but did not connect it to astronomical obs

• 1922 Ernst Öpik, by an ingenious method, had already found a distance of 450 kpc to Andromeda, much closer to the real value (775 kpc) than Hubble’s later distance of 285 kpc

• 1927 Lemaître derived value of $H$ using Hubble’s 1926 distances and Slipher’s redshifts; depending on choice of observations he arrived at 625 or 575 (compared to Hubble’s 500(km/s)/Mpc in 1929). He was fully aware of the significance of his discovery
• 1929 Hubble opted for $K = 500 \text{(km/s)}/\text{Mpc}$ as his favorite value, working with his own distances and Slipher’s redshifts, as tabulated in Eddington’s *The Mathematical Theory of Relativity* (2nd Ed 1924)

• Hubble refrained from interpreting his observational discovery: “The outstanding feature, however, is the possibility that the velocity-distance relation may represent the de Sitter effect...”. In a later letter to de Sitter, Hubble wrote he would leave the interpretation to those “competent to discuss the matter with authority”. In none of the 7 pages of Hubble’s paper is there a single word about an expanding universe; it is likely that Hubble never believed in such a thing

• It is not known when Einstein was converted to believe in the expanding Univ; it probably happened when Eddington showed him that his static solution was unstable

Stigler’s law of eponymy: ‘No scientific discovery is named after its original discoverer’ Kragh, Smith ‘03; Livio ‘11; Luminet, ‘11; Nussbaumer, Bieri, CUP ‘09 ‘11 ‘13
The accelerating Universe

Evidence for the acceleration of the Universe expansion:

- **distant supernovae**  AG Riess e a ’98, S Perlmutter e a ’99
- **baryon acoustic oscillations BAO**  Martin White [mwhite/bao/](http://mwhite/bao/)
- **the galaxy distribution**  AG Sanchez ea 12, Gaztanaga ea
- **correlations of galaxy distribs**  R Scranton ea ’03

Multiple sets of evidence: no systematics affect the conclusion that $\ddot{a} > 0$, $a$ scale factor of the Universe

- **the age of the Universe**  Planck Collab ’14
Zero point energy

QFT vacuum to vacuum transition: \[ \langle 0 | H | 0 \rangle \]

Spectrum, normal ordering (harm oscill):

\[
H = \left( n + \frac{1}{2} \right) \lambda_n \ a_n \ a_n^\dagger
\]

\[
\langle 0 | H | 0 \rangle = \frac{\hbar \ c}{2} \sum_n \lambda_n = \frac{1}{2} \ \text{tr} \ H
\]

gives \( \infty \) physical meaning?

Regularization + Renormalization (cut-off, dim, \( \zeta \ ))

Even then: Has the final value real sense?
The Riemann zeta function $\zeta(s)$ is a function of a complex variable, $s$. To define it, one starts with the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges for all complex values of $s$ with real $\text{Re} \ s > 1$, and then defines $\zeta(s)$ as the analytic continuation, to the whole complex $s$–plane, of the function given, $\text{Re} \ s > 1$, by the sum of the preceding series.

Leonhard Euler already considered the above series in 1740, but for positive integer values of $s$, and later Чебышёв extended the definition to $\text{Re} \ s > 1$. 
\[ \zeta(s) = \sum \frac{1}{n^s} \]
\( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \)

\( \zeta(0) = -\frac{1}{2} \) or \( 1+1+1+\cdots = -\frac{1}{2} \)

\( \zeta(-1) = -\frac{1}{12} \) or \( 1+2+3+\cdots = -\frac{1}{12} \)
The Riemann zeta function \( \zeta(s) \) is a function of a complex variable, \( s \). To define it, one starts with the infinite series
\[
\sum_{n=1}^{\infty} \frac{1}{n^s}
\]
which converges for all complex values of \( s \) with real \( \text{Re} \, s > 1 \), and then defines \( \zeta(s) \) as the analytic continuation, to the whole complex \( s \)-plane, of the function given, \( \text{Re} \, s > 1 \), by the sum of the preceding series.

Leonhard Euler already considered the above series in 1740, but for positive integer values of \( s \), and later Chebyshev extended the definition to \( \text{Re} \, s > 1 \).


Did much of the earlier work, by establishing the convergence and equivalence of series regularized with the heat kernel and zeta function regularization methods

G H Hardy, Divergent Series (Clarendon Press, Oxford, 1949)

Srinivasa I Ramanujan had found for himself the functional equation of the zeta function

Torsten Carleman, “Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes” (French), 8. Skand Mat-Kongr, 34-44 (1935)

Zeta function encoding the eigenvalues of the Laplacian of a compact Riemannian manifold for the case of a compact region of the plane
Robert T Seeley, “Complex powers of an elliptic operator. 1967

Extended this to elliptic pseudo-differential operators $A$ on compact
Riemannian manifolds. So for such operators one can define the
determinant using zeta function regularization

D B Ray, Isadore M Singer, “$R$-torsion and the Laplacian on
Riemannian manifolds”, Advances in Math 7, 145 (1971)

Used this to define the determinant of a positive self-adjoint operator
$A$ (the Laplacian of a Riemannian manifold in their application) with
eigenvalues $a_1, a_2, \ldots$, and in this case the zeta function is formally
the trace

$$\zeta_A(s) = \text{Tr} \ (A)^{-s}$$

the method defines the possibly divergent infinite product

$$\prod_{n=1}^{\infty} a_n = \exp[-\zeta_A'(0)]$$
Abstract

The effective Lagrangian and vacuum energy-momentum tensor $< T^{\mu\nu} >$ due to a scalar field in a de Sitter space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2 + \xi R + m^2)\varphi = 0$. If $\xi = 1/6$ and $m = 0$, the renormalized $< T^{\mu\nu} >$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1}$, where $a$ is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.
Effective Lagrangian and energy-momentum tensor in de Sitter space

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(Received 29 October 1975)

The effective Lagrangian and vacuum energy-momentum tensor \( \langle T^\mu \nu \rangle \) due to a scalar field in a de Sitter-space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form \((\xi^2 + \xi R + m^2)\phi = 0\). If \(\xi = 1/6\) and \(m = 0\), the renormalized \(\langle T^\mu \nu \rangle\) equals \(g^{\mu \nu}(960\alpha a^4)^{-1}\), where \(a\) is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

I. INTRODUCTION

In a previous paper\(^1\) (to be referred to as I) the effective Lagrangian \(\mathcal{L}^{(1)}\) due to single-loop diagrams of a scalar particle in de Sitter space was computed. It was shown to be real and was evaluated as a principal-part integral. The regularization method used was the proper-time one due to Schwinger\(^2\) and others. We now wish to consider the same problem but using different techniques. In particular, we wish to make contact with the work of Candelas and Raine,\(^3\) who first discussed the same problem using dimensional regularization.

Some properties of the various regularizations as applied to the calculation of the vacuum expectation value of the energy-momentum tensor have been contrasted by DeWitt.\(^4\) We wish to pursue some of these questions within the context of a definite situation.

II. GENERAL FORMULAS: REGULARIZATION METHODS

We use exactly the notation of I, which is more or less standard, and begin with the expression for \(\mathcal{L}^{(1)}\) in terms of the quantum-mechanical propagator, \(K(x'', x', \tau)\),

\[
\mathcal{L}^{(1)}(x') = \frac{1}{2} i \lim_{x'' \to x'} \int_0^\infty d\tau \tau^{-1} K(x'', x', \tau) e^{-i\pi \tau} + X(x')
\]

(1)

There are two points regarding this expression which need some further discussion. Firstly, if we adopt the proper-time regularization method so that the infinities appear only when the \(\tau\) integration, which is the final operation, is performed, then we can take the coincidence limit, \(x'' = x'\), through into the integrand. Further, since the regularized expression is continuous across the light cone, it does not matter how we let \(x''\) approach \(x'\). Secondly, the term \(X\) does not have to be a constant, but it should integrate to give a metric-independent contribution to the total action, \(\mathcal{W}^{(1)}\).

The Schwinger-DeWitt procedure is to derive an expression for \(K\), either in closed form or as a general expansion to powers of \(\tau\), then to effect the coincidence limit in (1), and finally to perform the \(\tau\) integration. This was the approach adopted in I. We proceed now to give a few more details.

We assume that we are working on a Riemannian space, \(\mathcal{M}\), of dimension \(d\). The coincidence limit \(K(x, x, \tau)\) can be expanded,\(^5\)

\[
K(x, x, \tau) = i(4\pi i \tau)^{-d/2} \sum_{n=0}^\infty a_n(x)(i \tau)^n,
\]

(2)

where the \(a_n\) are scalars constructed from the curvature tensor on \(\mathcal{M}\) and whose functional form is independent of \(d\). The manifold \(\mathcal{M}\) must not have boundaries, otherwise other terms appear in the expansion.

The expansion (2) is substituted into (1) to yield

\[
\mathcal{L}^{(1)}(x) = \frac{1}{2} i (4\pi)^{-d/2} \sum_{n=0}^\infty a_n(x) \int_0^\infty (i \tau)^{n-d/2} e^{-i\pi \tau} d\tau.
\]

(3)

The infinite terms are those for which \(n < d/2\) (for \(d\) even) or \(n < (d-1)/2\) (for \(d\) odd). For \(d=4\), e.g., space-time, there are three infinite terms. These terms are removed by renormalization; the details are given by DeWitt.\(^4\)

Another popular regularization technique is dimensional regularization.\(^6\) In this method the dimension, \(d\), is considered to be complex and all expressions are defined in a region of the \(d\) plane where they converge. The infinities appear when an analytic continuation to \(d=4\) is performed to regain the physical quantities. This idea was originally developed for use in flat-space (i.e., Lorentz-invariant) situations for the momentum

This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to $n$ dimensions by adding extra flat dims. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fifth dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This EM tensor has an anomalous trace.
Zeta Function Regularization of Path Integrals in Curved Spacetime

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Abstract. This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to $n$ dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fifth dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This energy momentum tensor has an anomalous trace.

1. Introduction

The purpose of this paper is to describe a technique for obtaining finite values to path integrals for fields (including the gravitational field) on a curved spacetime background or, equivalently, for evaluating the determinants of differential operators such as the four-dimensional Laplacian or D'Alambertian. One forms a generalised zeta function from the eigenvalues $\lambda_n$ of the operator

$$\zeta(s) = \sum_n \lambda_n^{-s}. \quad (1.1)$$

In four dimensions this converges for $\text{Re}(s) > 2$ and can be analytically extended to a meromorphic function with poles only at $s = 2$ and $s = 1$. It is regular at $s = 0$. The derivative at $s = 0$ is formally equal to $-\sum_n \log \lambda_n$. Thus one can define the determinant of the operator to be $\exp(-d\zeta/ds)|_{s=0}$. 
\[ Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \frac{1}{1 - \frac{1}{p^s}} \]

The prime number theorem

\[ \pi(x) = \# \{ \text{primes } p \leq x \} \sim \frac{x}{\log x} \]

\[ \xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \xi(s) = \xi(1-s) \]

(Completed $\zeta$-.)

(E) Entirety: \( \xi(s) \) meromorphic c., \( s = 0, 1 \) poles

(FE) \( \xi(s) = \xi(1-s) \)

(BV) Bounded in vertical strips:

\[ \xi(s) + \frac{1}{s} + \frac{1}{1-s} \text{ bounded } -\infty < a < \text{Re} s < b < \infty \]

Riemann (1859)

Poisson s. f.

\[ \sum_{n \in \mathbb{Z}} \hat{f}(n) = \sum_{n \in \mathbb{Z}} f(n) \]

\[ \hat{f}(r) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i r x} \, dx, \; \hat{f} \text{ Schwartz} \]

1) FE \( \Theta \)

2) \( \hat{f}(r) = \frac{1}{\sqrt{4\pi}} e^{-\pi r^2 / 4} \rightarrow \text{ Jacobi id.} \)

\[ \Theta(it) = \frac{1}{\sqrt{4\pi}} \Theta\left(\frac{t}{4\pi}\right), \; \Theta(t) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{\pi i n^2 t \frac{x}{2}} \]

Dirichlet \( L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \) Hamburger
Basic strategies

- **Jacobi’s identity** for the $\theta$–function

$$\theta_3(z, \tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \quad q := e^{i\pi \tau}, \quad \tau \in \mathbb{C}$$

$$\theta_3(z, \tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi \tau} \theta_3 \left( \frac{z}{\tau} \mid \frac{-1}{\tau} \right) \quad \text{equivalently:}$$

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}} \sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t^2}} \cos(2\pi nz), \quad z, t \in \mathbb{C}, \quad \text{Re} t > 0$$

- **Higher dimensions**: **Poisson summ formula** (Riemann)

$$\sum_{\vec{n} \in \mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} \tilde{f}(\vec{m}) \quad \tilde{f} \quad \text{Fourier transform}$$

[Gelbart + Miller, BAMS ’03, Iwaniec, Morgan, ICM ’06]

- **Truncated sums** $\longrightarrow$ **asymptotic series**
3: EXPLICIT CALCULATIONS

Epstein zeta functions (quadratic)
\[ Z_E = \sum_{\vec{n} \in \mathbb{Z}^d} \varphi(\vec{n})^{-s} \]

Barnes zeta functions (linear)
\[ Z_B = \sum_{\vec{n} \in \mathbb{N}^d} L(\vec{n})^{-s} \]

Extension:

\[ Z_E \quad \xrightarrow{\varphi + L \text{ affine}} \quad \sum_{\vec{n} \in \mathbb{N}^d} \]

\[ Z_B \quad \xrightarrow{Z'_{B}(0) \text{ (new formulas)}} \quad \sum'_{\vec{n} \in \mathbb{Z}^d} \]

(truncation)

(by analytic cut)
Example of the ball:

- Operator

\[ (-\Delta + m^2) \]

on the \( D \)-dim ball \( B^D = \{ x \in R^D; |x| \leq R \} \)

with Dirichlet, Neumann or Robin BC.

- The zeta function

\[ \zeta(s) = \sum_k \lambda_k^{-s} \]

- Eigenvalues implicitly obtained from

\[ (-\Delta + m^2)\phi_k(x) = \lambda_k \phi_k(x) + BC \]

- In spherical coordinates:

\[ \phi_{l,m,n}(r, \Omega) = r^{1-D/2} J_{l+D-2}(w_{l,n}r)Y_{l+D/2}(\Omega) \]

\( J_{l+(D-2)/2} \) Bessel functions

\( Y_{l+D/2} \) hyperspherical harmonics

- Eigenvalues \( w_{l,n} (> 0) \) determined through BC

\[ J_{l+D-2}(w_{l,n}R) = 0, \quad \text{for Dirichlet BC} \]
\[ \frac{u}{R} J_{l+ \frac{D-2}{2}} (w_{l,n} R) + w_{l,n} J'_{l+ \frac{D-2}{2}} (w_{l,n} r) \big|_{r=R} = 0, \text{ for Robin BC} \]

- Now, \( \lambda_{l,n} = w_{l,n}^2 + m^2 \)

\[ \zeta(s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} d_l(D) (w_{l,n}^2 + m^2)^{-s} \]

\( w_{l,n} (> 0) \) is defined as the n-th root of the l-th equation, \( d_l(D) = (2l + D - 2) \frac{(l+D-3)!}{l!(D-2)!} \)

- **Procedure:**
  - Contour integral on the complex plane

\[ \zeta(s) = \sum_{l=0}^{\infty} d_l(D) \int_{\gamma} \frac{dk}{2\pi i} (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln \Phi_{l+ \frac{D-2}{2}} (kR) \]

\( \gamma \) runs counterclockwise and must enclose all the solutions [Ginzburg, Van Kampen, EE + I. Brevik]

- **Obtained:** [with Bordag, Kirsten, Leseduarte, Vassilievich,...]

  - Zeta functions
  - Determinants
  - Seeley [heat-kernel] coefficients
Zeta functions on tori using contour integration

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Received 26 August 2013
Accepted 29 October 2014
Published 24 December 2014

A new, seemingly useful presentation of zeta functions on complex tori is derived by using contour integration. It is shown to agree with the one obtained by using the Chowla–Selberg series formula, for which an alternative proof is thereby given. In addition, a new proof of the functional determinant on the torus results, which does not use the Kronecker first limit formula nor the functional equation of the non-holomorphic Eisenstein series. As a bonus, several identities involving the Dedekind eta function are obtained as well.

Keywords: Tori; zeta functions; spectral theory; functional determinants.

Mathematics Subject Classification 2010: Primary 11M41, 81T40; Secondary 81T25

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Existence of $\zeta_A$ for $A$ a $\Psi$DO

1. $A$ a positive-definite elliptic $\Psi$DO of positive order $m \in \mathbb{R}^+$

2. $A$ acts on the space of smooth sections of $E$, $n$-dim vector bundle over $M$

3. $M$ closed $n$-dim manifold

(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \Re s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of $A$, $s_0 = \dim M/\text{ord } A$ abscissa of converg of $\zeta_A(s)$

(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of $A$, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

(c) The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$

(d) The only possible singularities of $\zeta^A(s)$ are poles at

$$s_j = (n - j)/m, \quad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$
**Definition of Determinant**

\[ H \Psi \text{DO operator} \{\varphi_i, \lambda_i\} \text{ spectral decomposition} \]

\[ \prod_{i \in I} \lambda_i \hspace{1cm} \ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i \]

Riemann zeta func: \[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \ Re \ s > 1 \] (& analytic cont)

Definition: zeta function of \( H \) \[ \zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr} \ H^{-s} \]

As Mellin transform: \[ \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} \text{tr} \ e^{-tH}, \ Re s > s_0 \]

Derivative: \[ \zeta'_H(0) = - \sum_{i \in I} \ln \lambda_i \]

Determinant: Ray & Singer, '67 \[ \text{det}_\zeta H = \exp [-\zeta'_H(0)] \]

Weierstrass def: subtract leading behavior of \( \lambda_i \) in \( i \), as \( i \to \infty \), until series \( \sum_{i \in I} \ln \lambda_i \) converges \( \implies \) non-local counterterms!!

C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...
Properties

The definition of the determinant $\det_{\zeta} A$ only depends on the homotopy class of the cut

A zeta function (and corresponding determinant) with the same meromorphic structure in the complex $s$-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C} \setminus \mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik]

Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A)t^{-s_j} + \sum_{k \geq 1} \beta_k(A)t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \text{Res}_{s=s_j} \zeta_A(s), \quad s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[ \text{PP } \zeta_A(-k) + \psi(k+1) \text{Res}_{s=-k} \zeta_A(s) \right],$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N} \setminus \{0\}$$

$$s_j = -k, \quad k \in \mathbb{N}$$

$$\text{PP } \phi := \lim_{s \to p} \left[ \phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]$$
"Hi, Emilio. This is a question I have been trying to solve for years. With a bit of luck you could maybe provide me with a hint or two.

- Imagine I've got a functional integral and I perform a point transformation (doesn't involve derivatives). Its Jacobian is a kind of functional determinant, but of a non-elliptic operator (it is simply infinite times multiplication by a function.) Did anybody study this seriously?

- I do know, from at least one paper I did with Luis AG, that in some cases (T duality) one is bound to define something like

$$\det f(x) \sim \det [f(x) O] / \det O$$

where $O$ is an elliptic operator (e.g. the Laplacian)

- This is what Schwarz and Tseytlin did in order to obtain the dilaton transformation

- And LAG and I did also proceed in a basically similar way

- As I know, Konsevitch, too, uses a related method involving the multiplicative anomaly

Tell me what you know about, please. Thanks so much.- Hugs, Enrique "
The Dixmier Trace

In order to write down an action in operator language one needs a functional that replaces integration.

For the Yang-Mills theory this is the Dixmier trace.

It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1, \infty)}$ of the compact operators $T$ such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \quad \mu_0 \geq \mu_1 \geq \cdots$$

Definition of the Dixmier trace of $T$:

$$\text{Dtr } T = \lim_{N \to \infty} \frac{1}{\log N} \log \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in $N$ are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^{\lambda} f(u) \frac{du}{u}$]

The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator $T^{-1}$ at $s = 1$ [Connes]

$$\text{Dtr } T = \lim_{s \to 1+} (s - 1) \zeta_{T^{-1}}(s)$$
The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to $\Psi$DOs which are not in $\mathcal{L}^{(1,\infty)}$.

- Only trace one can define in the algebra of $\Psi$DOs (up to multiplication constant).

- Definition: $\text{res } A = 2 \text{Res}_{s=0} \text{tr}(A\Delta^{-s})$, $\Delta$ Laplacian.

- Satisfies the trace condition: $\text{res } (AB) = \text{res } (BA)$.

- Important!: it can be expressed as an integral (local form)
  $$\text{res } A = \int_{S^*M} \text{tr } a_n(x,\xi) \, d\xi$$

  with $S^*M \subset T^*M$ the co-sphere bundle on $M$ (some authors put a coefficient in front of the integral: Adler-Manin residue).

- If $\dim M = n = -\text{ord } A$ ($M$ compact Riemann, $A$ elliptic, $n \in \mathbb{N}$), it coincides with the Dixmier trace, and $\text{Res}_{s=1} \zeta_A(s) = \frac{1}{n} \text{res } A^{-1}$.

- The Wodzicki residue makes sense for $\Psi$DOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace.
Given $A$, $B$, and $AB$ ψDOs, even if $\zeta_A$, $\zeta_B$, and $\zeta_{AB}$ exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$$
\[
\det_3(AB) = \det_3 A \det_3 B
\]
\[
\log \det_3 = \text{tr}_3 \log, \quad \det_3 = e^{\text{tr}_3 \log}
\]
\[
\det_3(AB) = e^{\text{tr}_3 \log(AB)} = e^{\text{tr}_3 (\log A + \log B)}
\]
\[
= e^{\text{tr}_3 \log A + \text{tr}_3 \log B}
\]
\[
= e^{\text{tr}_3 \log A} e^{\text{tr}_3 \log B}
\]
\[
= \det_3 A \cdot \det_3 B
\]

\[ [A, B] = 0 \quad \text{assumed!} \]

Which step is wrong?

\[ \text{tr}_3 \text{ is no trace at all} \]

\[ \text{tr}_3 (A + A_2) \neq \text{tr}_3 A + \text{tr}_3 A_2 \]

Recall
\[ \text{tr}_3 A = \mathbb{Z}_3 (5 = -1) = \sum_{n} \lambda_n \left| s = -1 \right| \]
Multipl or N-Comm Anomaly, or Defect

Given $A$, $B$, and $AB$ ψDOs, even if $\zeta_A$, $\zeta_B$, and $\zeta_{AB}$ exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \det_\zeta B$$

The multiplicative (or noncommutative) anomaly (defect) is defined as

$$\delta(A, B) = \ln \left[ \frac{\det_\zeta(AB)}{\det_\zeta A \det_\zeta B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

Wodzicki formula

$$\delta(A, B) = \frac{\text{res} \left\{ [\ln \sigma(A, B)]^2 \right\}}{2 \ord A \ord B (\ord A + \ord B)}$$

where

$$\sigma(A, B) = A^{\ord B} B^{-\ord A}$$
Consequences of the Multipl Anomaly

In the path integral formulation

\[ \int [d\Phi] \exp \left\{ - \int d^Dx \left[ \Phi^{\dagger}(x)( - ) \Phi(x) + \cdots \right] \right\} \]

Gaussian integration: \[ \rightarrow \quad \det \left( \begin{array}{c} \pm \end{array} \right) \]

\[ \left( \begin{array}{cc} A_1 & A_2 \\ A_3 & A_4 \end{array} \right) \rightarrow \left( \begin{array}{c} A \\ B \end{array} \right) \]

\[ \det(AB) \quad \text{or} \quad \det A \cdot \det B \quad ? \]

In a situation where a superselection rule exists, \( AB \) has no sense (much less its determinant): \[ \Rightarrow \det A \cdot \det B \]

But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: \[ \Rightarrow \det(AB) \]
Ten Physical Applications of Spectral Zeta Functions

Second Edition
Thank You

Спасибо

Gracias