On a Family of Non-local Gravity Models

E M I L I O   E L I Z A L D E

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Outline

- Intro: recalling a past story
- The accelerating Universe
- Different types of models
- Several problems
- Models with non-local interactions (Deser & Woodard): motivations
- Our model of type $f(\Box^{-1} R)$

With THANKS to:

Sergey Yu. Vernov, Ying-li Zhang, Ekaterina Pozdeeva, Sergei Odintsov, Misao Sasaki, Guido Cognola, Sergio Zerbini
SMITHSONIAN WASHINGTON
26 APRIL 1920

HARLOW SHAPLEY VS. HEBER CURTIS

GREAT DEBATE

WORLD HEAVYWEIGHT CHAMPIONSHIP
WBA / IBF / WBO / IBO

MOSCOW 2013
Apr 26, 1920: The Great Debate – Shapley vs Curtis

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The accelerating Universe

Dark Energy: effects on the expansion rate of Universe, 3 approaches:

— Standard candles: measure luminosity distance as function of redshift
— Standard rulers: angular diam distance & expansion rate as f of redsh
— Growth of fluctuations: generated at origin of U & amplified by inflation

Both angular diameter and luminosity distances are integrals of (inverse) expansion rate: encode effects of DE
To get competitive constraints on DE need see changes in $H(z)$ at 1%
would give statistical errors in DE EoS of $O(10\%)$

— calibrate the ruler accurately over most of the age of the universe
— measure the ruler over much of the volume of the universe
— make ultra-precise measurements of the ruler

On large scales or early times the perturbative treatment is valid:
calculations are perfectly under control
Length scales from physics of the early universe are imprinted on the
distribution of mass and radiation: they form time-independent rulers
(M White, Berkeley)
The accelerating Universe (II)

Evidence for the acceleration of the Universe expansion:

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- correlations of galaxy distribs
  R Scranton ea ’03,

Multiple sets of evidence: no systematics affect the conclusion that $\ddot{a} > 0$, a scale factor of the Universe
Vigorous star formation can blast gas out of a galaxy and starve future generations of stars of the fuel they need to form and grow. Enormous outflows of molecular gas are ejected by star-forming regions in the nearby Sculptor Galaxy NGC253 (11.5 MLY). They help explain the strange paucity of very massive galaxies in the Universe.
Different types of models

In General Relativity (GR): Gravity leads to deceleration

But pressure also influences geometry: R Tolman ’32

negative pressure can drive acceleration

Cosmological evidence could be explained by an undiscovered substance with negative pressure, so-called dark energy

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    negative pressure can drive acceleration
Cosmological evidence could be explained by an undiscovered
substance with negative pressure, so-called dark energy
J.A. Frieman ea '95, K. Coble ea '97, R. Caldwell ea '97, B. Ratra,
P.J.E. Peebles '98, C. Wetterich '98, D. Huterer, M.S. Turner '99

Another possib: GR is (wrong!) not accurate enough at large scales
S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner '04, Capozziello '04
GR was developed using information and intuition at Solar System
scales, it is (almost) only checked there, it could need to be modified
on large scales
A. Starobinsky arrived at same conclusion, through very different
arguments based on quantum corrections to ordinary GR: lead to
terms of second order in $R$, and higher
Several problems

Do not have simple guidelines, gedanken experiment, reasons of elegance and simplicity, as those of Einstein in constructing GR. Besides that, even if beauty is abandoned, a modification of gravity must still confront three additional problems (Park & Dodelson)
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Almost all models contain a mass scale to be set much smaller than any mass found in nature, $< 10^{-33}$ eV.

What is the meaning of this small mass scale?

How can it be protected from interactions with the rest of physics?
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Another problem: modified gravity models should comply with the successes of GR in the Solar System.

These constraints already doomed one of the first most promising modified gravity models introduced to explain acceleration and still place tight constraints on many models.
Models with non-local interactions

One class of modified gravity models that overcomes most of these problems contains non-local interactions


Deser and Woodard consider terms that are functionals of $\Box^{-1} R$

\(\Box\) the d’Alembertian and \(R\) the Ricci scalar
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At cosmological scales, $\Box^{-1} R$ grows very slowly:

- as $(t/teq)^{1/2}$ in the radiation dominated era
- logarithmically in the matter dominated era

So, at the time of Nucleosynthesis $\Box^{-1} R$ is about $10^{-6}$ and at matter-radiation equilibrium it is only order 1

In a natural way, these terms are irrelevant at early times and begin to affect the dynamics of the Universe only after the matter-radiation transition

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Finally, because $\Box^{-1}R$ is extremely small in the Solar System, these models easily pass local tests of gravity
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N. Koshelev, Grav. Cosmol. 15, 220 (2009), 0809.4927
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A realistic model for acceleration, with an arbitrary function of $R^{-1} \Box R$, reproducing the expansion history of ΛCDM

C. Deffayet, R. Woodard, JCAP 0908, 023 (2009), 0904.0961

Used by S. Park & S. Dodelson, arXiv:1209.0836, to discuss structure formation in a nonlocally modified gravity.
Additional considerations

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But a very interesting aspect of modified gravity models in general is that, even if they are constructed to reproduce the expansion history equivalent to that of a given dark energy model (such as $\Lambda\text{CDM}$), perturbations will often evolve differently than in a model with standard GR plus dark energy.
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But a very interesting aspect of modified gravity models in general is that, even if they are constructed to reproduce the expansion history equivalent to that of a given dark energy model (such as $\Lambda$CDM), perturbations will often evolve differently than in a model with standard GR plus dark energy.

Indeed, the way to distinguish DE models from modified gravity is to measure the growth of structure in the Universe. The deviations from dark energy models are at the 10 to 30% level and have a characteristic signature as a function of redshift, which suggests that the class of models could be tested by upcoming surveys.
Our model of type $f(\Box^{-1} R)$

Consider a nonlocal gravity which contains a function of the $\Box^{-1}$ operator, thus not assuming the existence of a new dimensional parameter in the action. We focus on the study of cosmological solutions both in Jordan and Einstein frames, including matter in the last case.
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Observationally, dark energy EoS parameter is now very close to $-1$, with tendency to stay below this value, what is intriguing. This number, if final, would lead back to GR with a cosmological constant (and nothing else). But a small deviation cannot be excluded by the most accurate astronomical data.
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Several models are able to reproduce observations, as quintom models, which involve two fields: a phantom and an ordinary scalar.

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We analyze the correspondence between solutions got in different frames and prove explicitly how knowledge of power-law solutions in Jordan's frame can be used to get power-law solutions in Einstein’s one.
The action

Consider a class of nonlocal gravities, with action

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ R \left( 1 + f(\Box^{-1} R) \right) - 2\Lambda \right] + \mathcal{L}_m \right\} \]

where \( \kappa^2 = 8\pi G = 8\pi / M_{Pl}^2 \), the Planck mass being \( M_{Pl} = G^{-1/2} = 1.2 \times 10^{19} \) GeV, \( f \) differentiable function (characterizes nature of nonlocality), \( \Box^{-1} \) inverse of d'Alembertian operator, \( \Lambda \) cosmological constant, and \( \mathcal{L}_m \) matter Lagrangian. For definiteness, we assume that matter is a perfect fluid. We use the signature \((-,-,+,+), \ g \) determinant of \( g_{\mu\nu} \)
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Introducing two scalar fields: \( \psi = \square^{-1} R \) & Lagrange multiplier \( \xi \)

\[ S_{loc} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[ R \left( 1 + f(\psi) \right) + \xi (R - \square \psi) - 2\Lambda \right] + \mathcal{L}_m \right\} \]

Original non-local action is recast as a local action in the Jordan frame. Varying this action with respect to \( \xi \) and \( \psi \), one resp gets the field eqs

\[ \square \psi = R , \quad \square \xi = f,\psi(\psi) R , \]

where \( f,\psi(\psi) \equiv df/d\psi \)
The action (II)

The corresponding Einstein equations are obtained by variation of the local action wrt the metric tensor

\[
\frac{g_{\mu\nu}}{2} \left[ R\Psi + \partial_\rho \xi \partial^\rho \psi - 2(\Lambda + \Box \Psi) \right] - R_{\mu\nu} \Psi - \frac{1}{2} \left( \partial_\mu \xi \partial_\nu \psi + \partial_\mu \psi \partial_\nu \xi \right) + \nabla_\mu \partial_\nu \Psi = -\kappa^2 T_{(m)\mu\nu}
\]

where \( \Psi \equiv 1 + f(\psi) + \xi \), and \( T_{(m)\mu\nu} \) energy-momentum tensor of matter sector

\[
T_{(m)\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_{m})}{\delta g^{\mu\nu}}
\]

Note the system of equations here does not include the function \( \psi \) itself, but instead \( f(\psi) \) and \( f,\psi(\psi) \), together with time derivatives of \( \psi \). Also, \( f(\psi) \) can only be determined up to a constant: one may add a constant to \( f(\psi) \) and subtract the same constant from \( \xi \) without changing eqs.
The action (II)

The corresponding Einstein equations are obtained by variation of the local action with respect to the metric tensor:

\[
g_{\mu\nu} \left[ R\Psi + \partial_\rho \xi \partial^\rho \psi - 2(\Lambda + \Box\Psi) - R_{\mu\nu} \Psi - \frac{1}{2} (\partial_\mu \xi \partial_\nu \psi + \partial_\mu \psi \partial_\nu \xi) + \nabla_\mu \partial_\nu \Psi \right] = \kappa^2 T(m)_{\mu\nu}
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Here, we assume a spatially flat FLRW universe:

\[
ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j
\]

and consider the case where the scalar fields \( \psi(t) \) and \( \xi(t) \) are functions of the cosmological time \( t \) only.
Thus, the system of Eqs. reduces to

\[
3H^2 \Psi = - \frac{1}{2} \dot{\xi} \dot{\psi} - 3H \dot{\Psi} + \Lambda + \kappa^2 \rho_m \\
\left(2\dot{H} + 3H^2\right) \Psi = \frac{1}{2} \dot{\xi} \dot{\psi} - \ddot{\Psi} - 2H \dot{\Psi} + \Lambda - \kappa^2 P_m \\
\dot{\psi} = -3H \dot{\psi} - 6 \left(\dot{H} + 2H^2\right) \\
\ddot{\xi} = -3H \ddot{\xi} - 6 \left(\dot{H} + 2H^2\right) f_{,\psi} (\psi)
\]

dot means differentiation with respect to time, \( t \), in the Jordan frame:

\( \dot{A}(t) \equiv \frac{dA(t)}{dt} \), \( H \equiv \frac{\dot{a}}{a} \) Hubble parameter

For a perfect matter fluid, \( T_{(m)\;00} = \rho_m \) and \( T_{(m)\;ij} = P_m g_{ij} \)
The action (III)

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\[ 3H^2 \Psi = -\frac{1}{2} \dot{\xi} \dot{\psi} - 3H \dot{\Psi} + \Lambda + \kappa^2 \rho_m \]

\[ \left( 2H + 3H^2 \right) \Psi = \frac{1}{2} \dot{\xi} \dot{\psi} - \ddot{\Psi} - 2H \ddot{\Psi} + \Lambda - \kappa^2 P_m \]

\[ \ddot{\psi} = -3H \dot{\psi} - 6 \left( \dot{H} + 2H^2 \right) \]

\[ \ddot{\xi} = -3H \dot{\xi} - 6 \left( \dot{H} + 2H^2 \right) f,\psi(\psi) \]

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The continuity equation is

\[ \dot{\rho}_m = -3H(P_m + \rho_m) \]

Adding them we obtain a 2nd order linear differential equation for \( \Psi \)

\[ \dddot{\Psi} + 5H \ddot{\Psi} + \left( 2\dot{H} + 6H^2 \right) \Psi - 2\Lambda + \kappa^2 (P_m - \rho_m) = 0 \]
Power-law sol’s with $f(\psi)$ exp funct

Consider the case when $f(\psi)$ exponential function

$$f(\psi) = f_0 e^{\alpha \psi}$$

$f_0$ and $\alpha$ nonzero real parameters. The motivation: (i) simplest model with power-law and de Sitter solutions (only exp or a sum of exps); (ii) better studied case among all possible functions for expanding universe sol’s (with Hubble parameter $H = n/t$, with $n$ a nonzero constant)
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We consider matter with the EoS parameter $w_m \equiv P_m/\rho_m$ being a constant but not equal to $-1$. For power-law solutions $H = n/t$, the continuity eq has the following general solution ($\rho_0$ arbitrary const)

$$\rho_m(t) = \rho_0 t^{-3n(w_m+1)}$$
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Solutions with $H = n/t$. Inserting $H = n/t$

\[ \psi(t) = \psi_1 t^{1-3n} - \frac{6n(2n-1)}{3n-1} \ln \left( \frac{t}{t_0} \right) \]

$\psi_1$, $t_0$ integration const’s. We consider real solutions at $t > 0$, hence, $t_0 > 0$. Note these valid provided $n \neq 1/3$ and $n \neq 1/2$ (special cases, other sect)
Summary of sol’s in the Jordan frame

Solutions in the Jordan frame for $m \neq 1 - 3n$, for $\Lambda = 0$ and for $\Lambda \neq 0$
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- Local constraints

  - **Newtonian limit** of the theory, described by corresponding local action check whether the power-law solutions found can satisfy this constraint
  - Depending on the integration constant $\xi_1$ being vanishing or not, we draw very different constraints on the Post-Newtonian parameter $\gamma$:
    - $\xi_1 \neq 0$, the constraint can be easily satisfied for a wide range of choices of the parameter $\alpha$
    - $\xi_1 = 0$ one needs to tune the parameter $\alpha$ to at least $10^{-5}$ order, to satisfy the local constraint

Note that, in previous papers, $\xi_1 = 0$ for simplicity. Analysis of the local constraint shows that solutions with nonzero $\xi_1$ allow to change the restrictions on the parameter $\alpha$, which are indeed necessary in order to make the model compatible with astronomical observations

- Power-law solutions for the original nonlocal model
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Varying the nonlocal action wrt the metric $g_{\mu\nu}$, under the spatially flat FLRW metric, the independent components of the field equations are

\[
3H^2 + \Delta G_{00} = \kappa^2 \rho_m + \Lambda, \quad -2\dot{H} - 3H^2 + \frac{1}{3\alpha^2} \delta^{ij} \Delta G_{ij} = \kappa^2 P_m - \Lambda
\]

$\Delta G_{00}$ and $\Delta G_{ij}$ are the modifications coming from the nonlocal terms

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\Delta G_{00} = \left[3H^2 + 3H \partial_t\right] \left\{ f\left(\Box^{-1} R\right) + \Box^{-1} \left[R \frac{df}{d(\Box^{-1} R)}\right]\right\}
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Identifying the scalar fields $\psi$ and $\xi$ with corresponding terms in original action

$$\psi(t) = \Box^{-1} R, \quad \xi(t) = \Box^{-1} \left[R \frac{df}{d(\Box^{-1} R)}\right]$$

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We conclude that these solutions are solutions of the initial nonlocal model as well, what can be checked immediately by direct substitution.
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Compared to the original form, the scalar-tensor presentation seems to have introduced a new degree of freedom $\chi$ (Woodard, Koivisto, 2010). In fact, if we write $\psi \rightarrow \psi + \chi$ the corresponding term is

$$\xi(\Box \psi - R) \rightarrow \xi(\Box(\psi + \chi) - R)$$

after integration by parts, change is $g^{\mu \nu} \partial_\mu \xi \partial_\nu \psi \rightarrow g^{\mu \nu} \partial_\mu \xi \partial_\nu (\psi + \chi)$.

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However, imposing appropriate BC, as $\chi = 0$, to recover original form, this would-be extra dof can be eliminated. Issue of choice of correct BC should be the only non-equivalence between original form and biscalalar-tensor representation. Thus, eg, Woodard ea determine the inverse d’Alembert op using the retarded Green function: they fix a solution of eq $\Box R = 0$ putting $\tilde{t}_0 = 0$ and $\eta_0 = 0$. 

A final comment. As stated above, the biscalar-tensor representation introduces two scalars, $\psi$ and $\xi$, therefore, working in this way it seems that one will encounter a ghost-like behavior (Koivisto, Nojiri, Bamba, Sasaki). However, since the original nonlocal model does not introduce any new degree of freedom, the ghost-like behavior of the biscalar-tensor theory may not be physically relevant: associated terms can be cast as a boundary term of the nonlocal operators (Koivisto 2010). At classical level, a necessary way to check this physical relevance is by considering the equivalence of the solutions coming from the original nonlocal formulation and from its biscalar-tensor form.
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Action and equations of motion in the Einstein frame
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Action and equations of motion in the Einstein frame

The Jordan and Einstein frames

Once a modified gravity theory is recast into its scalar-tensor presentation, it immediately follows that both the Jordan frame (where the matter sector minimally couples to gravity) and the Einstein one (where the Ricci is linear but matter couples to gravity non-minimally) are available. They are related by conformal transformation $g_{\mu\nu} = \Omega^2 g_{\mu\nu}^{(E)}$. Metric in Jordan frame is $g_{\mu\nu}$, in Einstein frame, $g_{\mu\nu}^{(E)}$. 

The conformal transformation connecting both frames cannot be simply interpreted as a coordinate transformation of the theory. This is the reason for long debate on which of the frames is ‘the physical one’ (the mathematical equivalence of the two frames is quite clear)
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First, formulate diff eqn for the conformal factor under which power-law sol’s in Jordan’s correspond to other power-law sol’s in Einstein’s.
In GR, power-law solutions of type $H = n/t$ correspond to models with a perfect fluid whose EoS param $w_m \equiv P_m/\rho_m$ is related to power-index by

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- Recasting original nonlocal action into local form, we obtained power-law sol's, with and without a cc, both in Jordan and Einstein frames. Showed that power-law sol's got in Jordan frame satisfy the original nonlocal equations. In the Jordan frame we proved that all power-law solutions are found that way
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In Einstein’s, we obtained the power-law solutions either by solving the EoM, or by performing a conformal transformation of the sol's obtained in Jordan’s. For this purpose, we extended the correspondence to include the matter sector. Using this powerful, non-trivial tool, we got the sol’s when $w_m = 1/3$ and $w_m = 1$ (very difficult to obtain by directly solving the system), thus proving the usefulness of the method.
We previously showed that not only models with \( \exp f(\psi) \) can have power-law and de Sitter solutions: a sum of \( \exp's \) too. Another generalization of this analysis is to include several perfect fluid components with different constant values of \( w_m \).
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From de Sitter solutions, interesting to check possibilities for Universe evolution as obtained from these nonlocal models, from inflationary dS stage to late power-law Universe. Check for deviations from standard GR case, its distinction from other modified gravities.
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The biscalar-tensor representation introduces two extra scalars. Can lead to a ghost problem. Equivalence between the initial nonlocal theory and local formulation has not been established yet. The ghost-like behavior of the biscalar-tensor theory may not be a physical problem, since the associated terms can be cast as boundary terms of the nonlocal operators (would-be ghost mode might correspond to an inappropriate choice of BC). Plan to consider this important question in future work.
An analysis of the stability of the solutions here encountered, of the ghost-free conditions, and of the cosmological perturbations corresponding to these models will be carried out next.
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Cosmological solutions of a nonlocal model with a perfect fluid

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