Scale Invariant Hard Thermal Loop resummation

Jean-Loïc Kneur (Lab. Charles Coulomb, Montpellier, France)

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Complete QCD phase diagram far from being confirmed:

\( T \neq 0, \mu = 0 \) well-established from lattice: no sharp phase transition, continuous crossover at \( T_c \approx 154 \pm 9 \text{ MeV} \)

Goal: more analytical approximations, ultimately in regions not much accessible on the lattice: large density (chemical potential) due to the famous “sign problem”

Tool: unconventional RG resummation of perturbative expansions

Very general: relevant both at \( T = 0 \) or \( T \neq 0 \) (and finite density too) → in particular addresses well-known problems of unstable + badly scale-dependent \( T \neq 0 \) perturbative expansions
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- Optimized (or 'screened') perturbation (OPT $\sim$ SPT)
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  (many similarities with QCD but simpler)
- Application: thermal (pure gauge) QCD: hard thermal loops
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Introduction/Motivations

Tool: unconventional 'RG-optimized' (RGOPT) resummation of perturbative expansions
Illustrate here $T \neq 0$ nonlinear $\sigma$-model, $+$ QCD (pure glue)

NB some previous results with our approach ($T = 0$):
estimate of the chiral symmetry breaking order parameter
$F_\pi(m_{u,d,s} = 0)/\Lambda_{\overline{MS}}^{QCD}$: $F_\pi$ exp input $\to \Lambda_{\overline{MS}}^{n_{f=3}} \to \alpha_{\overline{MS}}^S(\mu = m_Z)$.

$N^3LO$: $F_\pi^{m_q=0}/\Lambda_{\overline{MS}}^{n_{f=3}} \simeq 0.25 \pm 0.01 \to \alpha_S(m_Z) \simeq 0.1174 \pm 0.001 \pm 0.001$
(JLK, A.Neveu, PRD88 (2013))
(compares well with $\alpha_S$ lattice and world average values [PDG2016-17])

Also applied to $\langle \bar{q}q \rangle$ at $N^3LO$ (using spectral density of Dirac operator):
$\langle \bar{q}q \rangle^{1/3}_{m_q=0}(2 \text{GeV}) \simeq -(0.84 \pm 0.01)\Lambda_{\overline{MS}}$ (JLK, A.Neveu, PRD 92 (2015))
Parameter free determination! (compares well with latest lattice result)
Problems of thermal perturbative expansion (QCD, $g\phi^4$, ...)

known problem: poorly convergent and very scale-dependent (ordinary) perturbative expansions:

QCD (pure glue) pressure at successive (standard) perturbation orders shaded regions: scale-dependence for $\pi T < \mu < 4\pi T$
(illustration from Andersen, Strickland, Su ’10)
2. (Variationally) Optimized Perturbation (OPT)

Trick \(( T = 0)\): add and subtract a mass, consider \( m \delta \) as interaction:

\[ \mathcal{L}(g, m) \rightarrow \mathcal{L}(\delta g, m(1 - \delta)) \quad \text{(e.g. in QCD} \quad g \equiv 4\pi \alpha_s) \]

where \( 0 < \delta < 1 \) interpolates between \( \mathcal{L}_\text{free} \) and massless \( \mathcal{L}_\text{int} \);
\( \rightarrow m: \) arbitrary trial parameter

- Take any standard (renormalized) pert. series, expand in \( \delta \) after:
  \( m \rightarrow m (1 - \delta); \quad g \rightarrow \delta g \)
then \( \delta \rightarrow 1 \) (to recover original massless theory):

BUT a dependence in \( m \) remains at any finite \( \delta^k \)-order:
fixed typically by stationarity prescription: optimization (OPT):
\[ \frac{\partial}{\partial m} \text{(physical quantity)} = 0 \quad \text{for} \quad m = \tilde{m}_\text{opt}(\alpha_s) \neq 0: \]
- \( T = 0 \): exhibits dimensional transmutation: \( \tilde{m}_\text{opt}(g) \sim \mu e^{-\text{const.}/g} \)
- \( T \neq 0 \) similar idea: “screened perturbation” (SPT), or resummed “hard thermal loop (HTLpt)” (QCD): expand around quasi-particle mass.

Does this 'cheap trick' always work? and why?
Expected behaviour (ideally)

Not quite what happens, except in simple models:
- Convergence of this procedure for $D = 1 \phi^4$ oscillator (cancels large pert. order factorial divergences!) Guida et al ’95
- Particular case of ’order-dependent mapping’ Seznec, Zinn-Justin ’79
- QFT multi-loop calculations (specially $T \neq 0$) (very) difficult:
  $\rightarrow$ empirical convergence? not clear
- Main pb at higher order: OPT: $\partial_m(...) = 0$ has multi-solutions (some complex!), how to choose right one, if no nonperturbative “insight”??
3. RG compatible OPT (≡ RGOPT)

Main additional ingredient (JLK, A. Neveu '10):

Consider a physical quantity (perturbatively RG invariant)
e.g. in thermal context the pressure \( P(m, g, T) \):

in addition to: \[ \frac{\partial}{\partial m} P^{(k)}(m, g, \delta = 1) \big|_{m=\tilde{m}} = 0, \] (OPT)

Require (\(\delta\)-modified!) result at order \(\delta^k\) to satisfy (perturbative)
Renormalization Group (RG) equation:

\[
\text{RG} \left( P^{(k)}(m, g, \delta = 1) \right) = 0
\]

with standard RG operator:

\[
\text{RG} \equiv \mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma(g) \frac{\partial}{\partial m}
\]

\[
\beta(g) \equiv -b_0 g^2 - b_1 g^3 + \cdots, \quad \gamma(g) \equiv \gamma_0 g + \gamma_1 g^2 + \cdots
\]

→ Additional nontrivial constraint
\[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \] \[ P^{(k)}(m, g, \delta = 1) = 0 \]

Then using OPT AND RG completely fix \( m \equiv \bar{m} \) and \( g \equiv \bar{g} \).

But \( \Lambda_{\overline{\text{MS}}} (g) \) satisfies by def.:
\[ [\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}] \Lambda_{\overline{\text{MS}}} \equiv 0 \) (consistently at a given order for \( \beta(g) \)).

equivalent to:
\[ \frac{\partial}{\partial m} \left( \frac{P^k(m, g, \delta = 1)}{\Lambda_{\overline{\text{MS}}} (g)} \right) = 0 ; \quad \frac{\partial}{\partial g} \left( \frac{P^k(m, g, \delta = 1)}{\Lambda_{\overline{\text{MS}}} (g)} \right) = 0 \) for \( \bar{m}, \bar{g} \)

• Optimal \( \bar{m} \approx \Lambda_{\overline{\text{MS}}} (g) \), but true physical result from \( P(\bar{m}, \bar{g}, T) \)

• At \( T = 0 \) reproduces at first order exact nonperturbative results in simpler models [e.g. Gross-Neveu model]
**OPT + RG = RGOPT main features**

- **Usual OPT/Screened PT:** embarrassing freedom in interpolation trick: why not $m \rightarrow m(1 - \delta)^a$?
  
Most previous works ($T = 0$, Screened PT, HTLpt $T \neq 0$) do linear interpolation ($a = 1$) without deep justification but generally (we have shown) $a = 1$ spoils RG invariance!

- **OPT gives multiple $\bar{m}(g, T)$ solutions at increasing $\delta^k$-orders**
  
  → Our approach restores RG, requires optimal solution to match perturbation (i.e. Asymptotic Freedom for QCD ($T = 0$)): 
  
  $\alpha_S \rightarrow 0$ ($\mu \rightarrow \infty$):  $\bar{g}(\mu) \sim \frac{1}{2b_0 \ln \frac{\mu}{\bar{m}}} + \cdots$, $\bar{m} \sim \Lambda_{QCD}$
  
  → At successive orders AF-compatible optimal solution (often unique) *only* appears for universal critical $a$:
  
  $m \rightarrow m(1 - \delta)^{\frac{\gamma_0}{b_0}}$ (in general $\frac{\gamma_0}{b_0} \neq 1$)
  
  → RG consistency goes beyond simple “add and subtract” trick and removes any spurious solutions incompatible with AF

- **But does not always avoid complex OPT $\bar{m}$ solutions**
  (if these occur, possibly cured by renormalization scheme change)
Problems of thermal perturbation (QCD but generic)

Usual suspect: mix up of hard $p \sim T$ and soft $p \sim \alpha_S T$ modes.

Thermal 'Debye' screening mass $m_D^2 \sim \alpha_S T^2$ gives IR cutoff, BUT $\Rightarrow$ perturbative expansion in $\sqrt{\alpha_S}$ in QCD
$\rightarrow$ often advocated reason for slower convergence

Yet many interesting QGP physics features happen at not that large coupling $\alpha_S(\sim 2\pi T_c) \sim .5$, $(\alpha_S(\sim 2\pi T_c) \sim 0.3$ for pure glue)

Many efforts to improve this (review e.g. Blaizot, Iancu, Rebhan ’03):

Screened PT (SPT) (Karsch et al ’97) $\sim$ Hard Thermal Loop (HTLpt) resummation (Andersen, Braaten, Strickland ’99); Functional RG, 2PI formalism (Blaizot, Iancu, Rebhan ’01; Berges, Borsanyi, Reinosa, Serreau ’05)

Our RGOPT($T \neq 0$) essentially treats thermal mass 'RG consistently': $\rightarrow$ UV divergences induce mass anomalous dimension.

(NB some qualitative connections with recently advocated “massive scheme” approach (Blaizot, Wschebor ’14))
RGOPT ($T \neq 0$ generic example): $\phi^4$ (JLK, M.B Pinto, PRL116 ’16)

- Start from usual 2-loop PT free energy $m \neq 0, ~ T \neq 0$ ($\overline{\text{MS}}$ scheme):

$$ (4\pi)^2 F_0 = E_0 - \frac{m^4}{8} (3 + 4 \ln \frac{\mu}{m}) - \frac{T^4}{2} J_0 \left( \frac{m}{T} \right) + O(g)(2\text{-loop}) $$

$$ J_0 \left( \frac{m}{T} \right) \sim \int_0^\infty dp \frac{1}{\sqrt{p^2 + m^2}} \frac{1}{e^{\sqrt{p^2 + m^2}} - 1} $$

- $F(T = 0)$ has LO $\ln \mu$ dependence: compensated by $E_0$ finite

  T-independent 'vacuum energy' subtraction (well-known at $T = 0$):

$$ E_0 (g, m) = -m^4 \left( \frac{s_0}{g} + s_1 + s_2 g + \cdots \right) $$

determined such that $\mu \frac{d}{d \mu} E_0$ cancels the $\ln \mu$ dependence:

$$ s_0 = \frac{1}{2(b_0 - 4\gamma_0)} = 8\pi^2; \quad s_1 = \frac{(b_1 - 4\gamma_1)}{8\gamma_0 (b_0 - 4\gamma_0)} = -1, \ldots $$

- Missed by SPT, HTLpt (QCD): explains the large scale dependence observed at higher order in those approaches! (more on this below)

- Next: expand in $\delta$, $\delta \rightarrow 1$ after $m^2 \rightarrow m^2 (1 - \delta)^a$ ; $g \rightarrow \delta g$

  RG only consistent for $a = 2\gamma_0 / b_0$ (= 1/3 for $\phi^4$ while $a = 1$ in SPT)

  NB: $1/g$ in $E_0$ automatically cancels in optimized energy $\mathcal{F}(\tilde{m})$.

- All together lead to a much better RGOPT residual scale dependence

  (factor $\sim 3$ better at 2-loops wrt PT/SPT, much better at 3-loops)
4. Closer to QCD: 2D $O(N)$ nonlinear $\sigma$ model (NLSM)

- Shares properties with QCD (asymptotic freedom, mass gap).
- At $T \neq 0$ the pressure, trace anomaly, etc have QCD-like shapes.
- Nonperturbative $T \neq 0$ results available for comparison: (lattice ($N = 3$)[Giacosa et al ’12], 1/$N$-expansion [Andersen et al ’04])

\[
\mathcal{L}_0 = \frac{1}{2} (\partial \pi_i)^2 + \frac{g (\pi_i \partial \pi_i)^2}{2(1 - g \pi_i^2)} - \frac{m^2}{g} (1 - g \pi_i^2)^{1/2}
\]

Two-loop pressure from: 

![Diagram of two-loop pressure](chart)

- Advantage w.r.t. QCD: exact $T$-dependence at 2-loops:

\[
P_{\text{pert,2loop}} = -\frac{(N - 1)}{2} \left[ l_0(m, T) + \frac{(N - 3)}{4} m^2 g l_1(m, T)^2 \right] + \mathcal{E}_0,
\]

\[
l_0(m, T) = \frac{1}{2\pi} \left( m^2 (1 - \ln \frac{m}{\mu}) + 4 T^2 K_0(\frac{m}{T}) \right)
\]

\[
K_0(x) = \int_0^{\infty} dz \ln \left( 1 - e^{-\sqrt{z^2 + x^2}} \right), \quad l_1(m, T) = \partial l_0(m, T)/\partial m^2
\]
One-loop RGOPT for NLSM pressure

Exact \( T \)-dependent mass gap \( \tilde{m}(g, T) \) from \( \partial_m P(m) = 0 \):

\[
\ln \frac{\tilde{m}}{\mu} = - \frac{1}{b_0 g(\mu)} - 2K_1\left( \frac{\tilde{m}}{T} \right), \quad (b_0^{\text{nlsm}} = \frac{N - 2}{2\pi})
\]

- Exhibits exact (one-loop) scale invariance:

\( T = 0 \): \( \tilde{m}(T = 0) = \mu e^{- \frac{1}{b_0 g(\mu)}} = \Lambda^{\text{1-loop}}_{\text{MS}} \)

\( T \gg m \):

\[
\frac{\tilde{m}(T)}{T} = \frac{\pi b_0 g}{1 - b_0 g \, L_T}, \quad (L_T \equiv \ln \frac{\mu e^{\gamma_E}}{4\pi T})
\]

\[
P_{1L, \text{exact}}^{\text{RGOPT}} = - \frac{(N - 1)}{\pi} T^2 \left[ K_0(\bar{x}) + \frac{\bar{x}^2}{8} (1 + 4K_1(\bar{x})) \right], \quad (\bar{x} \equiv \tilde{m}/T)
\]

\[
P_{1L}^{\text{RGOPT}}(T \gg m) \approx 1 - \frac{3}{2} b_0 g\left( \frac{4\pi}{e^{\gamma_E}} \right) T
\]

with one-loop running \( g^{-1}(\mu) = g^{-1}(M_0) + b_0 \ln \frac{\mu}{M_0} \)

\( \rightarrow \) nice RGOPT property: running with \( T \) emerges consistently (for standard perturbation, SPT, HTLpt, \( \mu \sim 2\pi T \) 'chosen')
NLSM pressure [G. Ferrari, JLK, M.B. Pinto, R.O. Ramos, 1709.03457, PRD]

\[ P/P_{SB}(N = 4, g(M_0) = 1) \] vs standard perturbation (PT), large N (LN), and SPT \( \equiv \) ignoring RG-induced subtraction; \( m^2 \to m^2(1 - \delta) \):

\[ P_{SB} = \frac{\pi}{6}(N - 1)T^2 \]

(Shaded range: scale-dependence \( \pi T < \mu \equiv M < 4\pi T \))

\( \to \) At two-loops a moderate scale-dependence reappears, although less pronounced than 2-loop standard PT, SPT.

Higher order: RGOPT at \( O(g^k) \) \( \to \) \( \bar{m}(\mu) \) appears at \( O(g^{k+1}) \) for any \( \bar{m} \), but \( \bar{m} \sim gT \to P \sim \bar{m}^2/g + \cdots \) has leading \( \mu \)-dependence at \( O(g^{k+2}) \).
NLSM interaction measure (trace anomaly)

(normalized) \( \Delta_{\text{NLSM}} \equiv (\mathcal{E} - P) / T^2 \equiv T \partial_T \left( \frac{P}{T^2} \right) \)

\( N = 4, g(M_0) = 1 \) (shaded regions: scale-dependence \( \pi T < \mu = M < 4\pi T \))

\( \bullet \) 2-loop \( \Delta_{\text{SPT}} \): small, monotonic behaviour + sizeable scale dependence.

\( \bullet \) RGOPT shape 'qualitatively' comparable to QCD, showing a peak: only obtained from interplay between \( T \neq 0 \) and \( T = 0 \) nonperturbative mass gap.

(But no phase transition in 2D NLSM (Mermin-Wagner-Coleman theorem): just reflects broken conformal invariance (mass gap)).
5. Thermal (pure glue) QCD: hard thermal loop (HTL)

QCD(glue) adaption of OPT → HTLpt [Andersen, Braaten, Strickland ’99]:
same trick now operates on a gluon mass term [Braaten-Pisarski ’90]:

\[ \mathcal{L}_{QCD}^{\text{gauge}} - \frac{m^2}{2} \text{Tr} \left[ G_{\mu\alpha} \langle \frac{y^\alpha y^\beta}{(y \cdot D)^2} \rangle y G^\mu_\beta \right], \quad D^\mu = \partial^\mu - ig A^\mu, \quad y^\mu = (1, \vec{y}) \]

(effective, explicitly gauge-invariant but nonlocal Lagrangian):

originally describes screening mass \( m^2 \sim \alpha_S T^2 + \text{other HTL contributions} \) [dressing gluon vertices and propagators]

But here \( m \) is arbitrary: determined by optimization in RGOPT.

\[
P_{\text{HTL, exact}}^{\text{1-loop}} = (N_c^2 - 1) \times \\
\left\{ \frac{m^4}{64\pi^2} (C_{11} - \ln \frac{m}{\mu}) + \int_0^\infty \frac{d\omega}{2\pi^3} \frac{1}{e^{\frac{\omega}{T}} - 1} \int_\omega^\infty dk \, k^2 (2\phi_T - \phi_L) \\
- \frac{T}{2\pi^2} \int_0^\infty dk \, k^2 \left[ 2 \ln(1 - e^{-\frac{\omega}{T}}) + \ln(1 - e^{-\frac{\omega}{T}}) \right] - \frac{\pi^2 T^4}{90} \right\}
\]

where \( k^2 + m^2 \left[ 1 - \frac{\omega_L}{2k} \ln(\frac{\omega_L + k}{\omega_L - k}) \right] = 0; \ f(\omega_T) = 0; \ \phi_L, \phi_T: \text{complicated.} \)

- Exact 2-loop? daunting task...

NB possibly simpler effective gluon mass models/prescriptions exist...
[e.g. Reinosa, Serreau, Tissier, Weschbor ’15]
...But HTLpt advantage: calculated up to 3-loops $\alpha_S^2$ (NNLO)

at two-loops:

[Andersen et al ’99–’15] BUT only as $m/T$ expansions

Drawback: HTLpt $\equiv$ high-$T$ approximation (by definition)

$$P_{\text{HTLpt}}^{1\text{-loop,}\overline{\text{MS}}} = P_{\text{ideal}} \times$$

$$\left[ 1 - \frac{15}{2} \hat{m}^2 + 30 \hat{m}^3 + \frac{45}{4} \hat{m}^4 \left( \ln \frac{\mu}{4\pi T} + \gamma_E - \frac{7}{2} + \frac{\pi^2}{3} \right) \right]$$

$$\hat{m} \equiv \frac{m}{2\pi T} \quad P_{\text{ideal}} = \left( N_c^2 - 1 \right) \pi^2 \frac{T^4}{45}$$
Standard HTLpt results

Pure gauge at NNLO (3-loops) [Andersen, Strickland, Su '10]:

![Plot of pressure vs. \(T/T_c\)](image)

Reasonable agreement with lattice results (down to \(T \sim 2 - 3 T_c\)) only emerges at NNLO (3-loop) for low scale \(\mu \sim \pi T - 2\pi T\).

NB even better agreement with lattice (for central scale choice) when including quarks.

Main HTLpt issue: drastically increasing scale dependence at NNLO order

Moreover HTLpt mass prescription: \(\bar{m} \rightarrow m_D^{\text{pert}}(\alpha_S)\) [rather than optimizing \(\partial_m P(m)\), to avoid complex optimized solutions]:

but OPT generally captures “more nonperturbative” information.
RGOPT adaptation of HTLpt = RGOHTL

Main changes:
• Crucial RG invariance-restoring subtractions in Free energy (pressure): 
  \[ P_{HTLpt} \to P_{HTLpt} - m^4 \left( \frac{s_0}{\alpha_s} + s_1 + \cdots \right) \]: reflects its anomalous dimension.
• Interpolate with \( m^2 (1 - \delta) \frac{\gamma_0}{b_0} \), where gluon mass anomalous dimension defined from (available) counterterm.

• Scale dependence improves at higher orders since RG invariance maintained at all stages:
  - from subtraction terms (prior to interpolation)
  - from interpolation keeping RG invariance.

• SPT, HTLpt, ... do not fulfill this:
  yet harmless (scale dependence moderate) up to 2-loops, because the (leading order) RG-unmatched term \( \mathcal{O}(m^4 \ln \mu) \) is perturbatively ‘3-loop’ \( \mathcal{O}(\alpha_s^2) \) from \( m^2 \sim \alpha_s T^2 \).

→ But explains why HTLpt scale dependence dramatically resurfaces at 3-loops!
Preliminary RGO(HTL) results (1- and 2-loop, pure glue)

One-loop: exactly scale-invariant pressure (like $\phi^4$ and NLSM):

$$\frac{P(T \gg m)}{P_{\text{ideal}}} = 1 - \frac{15}{4} \hat{m}^2 + \frac{15}{2} \hat{m}^3 + \mathcal{O}(\hat{m}^6)$$

$$\hat{m}(OPT) = G \left(1 + \sqrt{1 - \frac{1}{3G}}\right), \quad G \text{'coupling'} = (\ln \frac{4\pi T}{\Lambda_{\overline{\text{MS}}}} + \text{const.})^{-1}$$

• Once accepting arbitrary $m$ in $\mathcal{L}_{\text{HTL}}$, like in NLSM RGOPT includes nontrivial $P(T = 0) \simeq -\text{const} \Lambda_{\overline{\text{MS}}}^4$

• Pb however: this 'exact' OPT $\hat{m}$ becomes complex for small enough $G$: essentially an artefact of $\overline{\text{MS}}$-scheme + high-$T$ approx
  (gives small $\text{Im}[P]/\text{Re}[P]$)

• Our attitude: one-loop approximation is not final stage:
  Pragmatic: at one-loop we thus take $\text{Re}[P(g)]$ in relevant $T/T_c$ range.

• Yet consistent with Stefan-Boltzmann limit: $P(g \to 0) \to P_{\text{ideal}}$

Alternatively evade this pb if adopting HTLpt prescription: $m \to m_{\text{Debye}}^{PT}$
  (consistent with standard PT: $P/P_{SB} = 1 - 15/4(\alpha_S/\pi) + \cdots$)
  but loose exact 1-loop scale invariance

• 2-loops: RGOPT gives a real unique solution.
RGOPT vs HTLpt: one-loop pressure

![Graph showing comparison between RGOPT and HTLpt one-loop pressure]

bands: $\pi T < \mu < 4\pi T$

NB: bending of $P_{\text{RGOPT}}$ for small $T$ essentially due to $-P_{\text{RGOPT}}(T = 0) \neq 0$. 
2-loop RGOHTL: need new calculations...

Crucial RG-consistent 2-loop subtractions determined by $\alpha_S m^4 \ln \mu$ term (non-logarithmic terms relevant too).

But these are $O(\alpha_S T^4 \frac{m^4}{T^4})$ of 2-loop HTL contributions, not available in literature (to best of our knowledge):
give $\sim 30$ independent integrals, half being (very) complicated, e.g.

$$\int_{P,Q} \frac{T_P T_Q (p + q)^2}{p^2 p^2 q^2 Q^2 (P + Q)^2}; \quad T_P \equiv \int_0^1 dc \frac{P_0^2}{P_0^2 + c^2 p^2}$$

($P^2 = P_0^2 + p^2$), $c \equiv$ HTL angle (averaging).

Present status: work under good progress but need to be checked,
specially the difficult non-logarithmic parts (i.e. finite parts in dim.reg.)

NB high- $T \leftrightarrow T = 0$ correspondance:

$C_{20} \ln^2 \frac{\mu}{T} + C_{21} \ln \frac{\mu}{T} + C_{22}^{(T \gg m)} \leftrightarrow C_{20} \ln^2 \frac{\mu}{m} + C_{21} \ln \frac{\mu}{m} + C_{22}^{(T = 0)}$

• However Leading Log (LL) $C_{20}$ determined straightforwardly from RG from one-loop.

• Similarly the 2-loop (perturbative) scale invariance is guaranteed by

$s_1 = f[C_{10}]$ independently of precise $C_{10}$ value!
(Preliminary!) RGO(HTL) results (2-loop, pure glue)

2-loop illustrated here for simplest LL approx.: $C_{21} = C_{22} = 0$
but RG-subtraction $s_1(C_{21} = 0)$ consistent

Moderate scale-dependence reappears at 2-loops
but sensible improvement wrt HTLpt

\[ \frac{P}{P_{ideal}} \]

[HTLpt 2loop]
[RG OPT 2loop]
[HTLpt 3loop]

Lattice (Boyd et al '96)

$\pi T < \mu < 4\pi T$

\[ \frac{T}{T_c} \]

[NB scale dependence should further improve at 3-loops, generically:

RG OPT at $\mathcal{O}(\alpha_S^k) \rightarrow \bar{m}(\mu)$ appears at $\mathcal{O}(\alpha_S^{k+1})$ for any $\bar{m}$, but

$\bar{m}^2 \sim \alpha_S T^2 \rightarrow P \sim \bar{m}_G^4 / \alpha_S + \cdots$: leading $\mu$-dependence at $\mathcal{O}(\alpha_S^{k+2})$.

• Warning: low $T \sim T_c$ genuine $P(T)$ shape sensitive to true $C_{21}, C_{22}$
(crucially needed before possibly comparing RGOPT vs lattice).]
(Very) preliminary RGO(HTL) approximate 3-loop results

3-loops: exact $m^4 \alpha_S^2$ terms need extra complicated calculations, but
$P^{3l}_{RGOHTL} \sim P^{2l}_{RGOHTL} + m^4 \alpha_S^2 \left( C_{30} \ln^3 \frac{\mu}{2\pi T} + C_{31} \ln^2 \frac{\mu}{2\pi T} + C_{32} \ln \frac{\mu}{2\pi T} + C_{33} \right)$:

leading logarithms (LL) and next-to-leading (NLL) $C_{30}, C_{31}$ fully
determined from lower orders from RG invariance

Within this LL, NLL approximation and in $T/T_c \gtrsim 2$ range where it is
more trustable:

![Graph showing the ratio of $\Delta P / P$ vs. $T/T_c$ for different approximations.]

We assume/expect true coefficients will not spoil this improved scale
dependence.
Summary and Outlook

- **RG OPT** includes 2 major differences w.r.t. previous OPT/SPT/HTLpt... approaches:
  1) OPT +/- or RG optimizations fix optimal $\bar{m}$ and possibly $\bar{g} = 4\pi \alpha_s$
  2) Maintaining RG invariance uniquely fixes the basic interpolation $m \to m(1 - \delta)^{\gamma_0/b_0}$: discards spurious solutions and accelerates convergence.

- At $T \neq 0$, exhibits improved stability + drastically improved scale dependence (with respect to standard PT, but also w.r.t. HTLpt)

- Paves the way to extend such RG-compatible methods to full QCD thermodynamics (work in progress, starting with $T \neq 0$ pure gluodynamics) specially for exploring also finite density