ICE CSIC IEEC

Zeta functions, heat kernels, and the quantum vacuum: some non-standard cases

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Heat Kernels 06, Blaubeuren, November 30, 2006

Outline of this presentation

- ΨDOs, Zeta Functions, Determinants, and Traces

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- Relation with regularized products

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m: M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \{f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha,\beta \in \mathbb{N}^n\}$$

- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

An elliptic ΨDO is one with an elliptic symbol

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ΨDOs are basic tools both in Mathematics & in Physics:

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]
2. Proof of the Atiyah-Singer index formula
3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]
4. Constitute nowadays the basic starting point of any rigorous formulation of QFT field theory through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions) [Fredenhagen, Rehren, Seiler, Wald, Brunetti, Verch, Radzikowski, ...]

[RW] The history and present status of QFT in curved spacetime, gr-qc/0608018

[F+R+S] QFT: where we are, hep-th/0603155 (Ans. Interact QFT at perturb level)

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as $\zeta_A(s) = \mathrm{tr}\, A^{-s} = \sum_j \lambda_j^{-s}$, Re $s > \frac{n}{m} \equiv s_0$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A ($a_m(x, \xi)$) admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_k = (n-k)/m, \qquad k = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

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Definition: zeta function of H $\zeta_H(s) = \sum_{i \in I} \lambda_i^{-s} = \text{tr } H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \text{tr } e^{-tH}, \quad Re\, s > s_0$

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Weierstrass definition: subtract leading behavior of λ_i in i, as $i \to \infty$, until the series $\sum_{i\in I} \ln \lambda_i$ converges

$\Longrightarrow$ non-local counterterms !!

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- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum\nolimits'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum\nolimits_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum\nolimits_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j)\text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[PP\zeta_A(-k) + \psi(k+1)\text{Res}_{s=-k} \zeta_A(s) \right],$$

$$s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$PP\phi = \lim\nolimits_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]$$

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- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

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$$\text{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

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- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [Connes]

$$\text{Dtr } T = \lim_{s \to 1^+} (s-1) \zeta_{T^{-1}}(s)$$

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$$\text{res}\ A = \int_{S^*M} \text{tr}\ a_n(x,\xi)\ d\xi$$

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- The Wodzicki res makes sense for ΨDOs of arbitrary order. Even if symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

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- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x,\xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1}=0$:

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- Proof. Homog component of degree $-n$ of the corresp power of the principal symbol of A are obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial \xi^{-1}}\right)^k \left[\xi^{n-k} a^{(k-n)/m}(x,\xi)\right]\Big|_{\xi^{-1}=0} \quad \xi^{-n}$$

Multipl or N-Comm Anomaly, or Defect

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- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2 \text{ ord } A \text{ ord } B \,(\text{ord } A + \text{ord } B)}$$

where $\sigma(A,B) = A^{\text{ord}\, B} B^{-\text{ord}\, A}$

Consequences of the Multipl Anom

- In the path integral formulation

$$\int [d\Phi] \, \exp\left\{ - \int d^D x \left[\Phi^\dagger(x) (\quad) \Phi(x) + \cdots \right] \right\}$$

Gaussian integration: $\longrightarrow \quad \det(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

$\det(AB)$ or $\det A \cdot \det B$?

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$\det(AB)$ or $\det A \;\cdot\; \det B$?

- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$

Consequences of the Multipl Anom

- In the path integral formulation

$$\int [d\Phi] \; \exp\left\{ - \int d^D x \left[\Phi^\dagger(x) (\quad) \Phi(x) + \cdots \right] \right\}$$

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$\det(AB)$ or $\det A \cdot \det B$?

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- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

The Chowla-Selberg Formula (CS)

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History

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$$\sum_{\lambda=1}^{|D|} \left(\frac{D}{\lambda}\right) \log\Gamma\left(\frac{\lambda}{D}\right) = h\log|D| - \frac{h}{3}\log(2\pi) - \sum_{(a,b,c)} \log a + \frac{2}{3}\sum_{(a,b,c)} \log\left[\theta_1'(0|\alpha)\theta_1'(0|\beta)\right]$$

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- Eta evaluations Dedekind eta function for Im $(\tau) > 0$

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q := e^{2\pi i \tau}$$

  It is a 24-th root of the discriminant func $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \,|\, a, b \in \mathbb{Z}\}$

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

Properties & Recent Results

$\Longrightarrow$ The C-S formula gives the value of a product of eta functions

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Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau \in \mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

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$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \qquad z,t \in \mathbb{C},\ Re\,t > 0$$

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$\tilde f$ Fourier transform [Gelbart + Miller, BAMS, Oct. '03]

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- Truncated sums

$\longrightarrow$ asymptotic series

Extended CS Formulas (ECS)

- Consider the zeta function (Re $s > p/2,\ A > 0$, Re $q > 0$):

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}(\vec{n}+\vec{c})^T A\,(\vec{n}+\vec{c}) + q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\,(\vec{n}+\vec{c}) + q\right]^{-s}$$

prime: point $\vec{n} = \vec{0}$ to be excluded from the sum
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- Case $q \neq 0$ (Re $q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}}\,\frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\;\Gamma(s)}$$

$$\times \sum_{\vec{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c})\,\left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

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[ECS1]

- Pole: $s = p/2$ Residue:

$$\text{Res}_{s=p/2}\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}$$

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one can select those vectors in $\mathbb{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive]

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- Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s}\, \Gamma\left(s - \frac{j}{2}\right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right)\right]$$

[ECS3d]

QFT in s-t with non-comm toroidal part

- $D-$dim non-commut manifold: $M = \mathbb{R}^{1,d} \bigotimes \mathbb{T}^p_\theta, \quad D = d + p + 1$
  $\mathbb{T}^p_\theta$ a $p-$dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$
  σ_{jk} a real, nonsingular, antisymmetric matrix of ± 1 entries
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  [Connes,Douglas,Seiberg,Cheung,Chu,Chomerus,Ardalan, . . .]

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- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s) = \frac{V\,\Gamma(s-(d+1)/2)}{(4\pi)^{(d+1)/2}\,\Gamma(s)} \sum_{\vec{n}\in\mathbb{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s} \left[1+\Lambda\theta^{2-2\alpha}Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s}$$

$\alpha = 2$ bos, $\alpha = 3$ ferm, $V = \text{Vol}\,(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j = a_j^{-1/2}$ compactific radii

- After some calculations,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} (-\Lambda\theta^{2-2\alpha})^l \, \zeta_{Q,\vec{0},0}(s+\alpha l - \frac{d+1}{2})$$

for all radii equal to R, with $I(\vec{n}) = \sum_{j=1}^{p} n_j^2$,

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where we use the notation $\zeta_E(s) \equiv \zeta_{I,\vec{0},0}(s)$

e.g., the Epstein zeta function for the standard quadratic form

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- Rich pole structure: pole of Epstein zf at $s = p/2 - \alpha k + (d+1)/2 = D/2 - \alpha k$, combined with poles of Γ, yields a rich pattern of singul for $\zeta_\alpha(s)$

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- Classify the different possible cases according to the values of d and $D = d + p + 1$. We obtain, at $s = 0$,

For $d = 2k$
$$\begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = 0 \\ \text{if } D = \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \end{cases}$$

For $d = 2k - 1$
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Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means multiple of 2α)

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Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means multiple of 2α)

$\Longrightarrow$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$,
& specific pole structure

$$\zeta_\alpha(s) = \frac{2^{s-d}\,V}{(2\pi)^{(d+1)/2}\Gamma(s)}\sum_{l=0}^{\infty}\frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)}(-2^\alpha\Lambda\theta^{2-2\alpha})^l\sum_{j=0}^{p-1}(\det A_j)^{-\frac{1}{2}}$$

$$\times\Big[\pi^{j/2}a_{p-j}^{-s-\alpha l+(d+j+1)/2}\Gamma(s+\alpha l-(d+j+1)/2)\zeta_R(2s+2\alpha l-d-j-1)$$

$$+4\pi^{s+\alpha l-(d+1)/2}a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4}\sum_{n=1}^{\infty}\sum_{\vec{m}_j\in\mathbf{Z}^j}{}' n^{(d+j+1)/2-s-\alpha l}$$

$$\times\left(\vec{m}_j^tA_j^{-1}\vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4}K_{(d+j+1)/2-s-\alpha l}\left(2\pi n\sqrt{a_{p-j}\vec{m}_j^tA_j^{-1}\vec{m}_j}\right)\Big]$$

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p \ D	even	odd
odd	(1a) pole / finite ($l\geq l_1$)	(2a) pole / pole
even	(1b) double pole / pole ($l\geq l_1, l_2$)	(2b) pole / double pole ($l\geq l_2$)

General pole structure of $\zeta_\alpha(s)$, for the possible values of D and p being odd or even. Magenta, type of behavior corresponding to lower values of l; behavior in blue corresponds to larger values of l

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- One-loop approx in QFT: Euclidean 1ℓ effective action is sum of classical action and a functional determinant of an elliptic diff op: the fluctuation operator (needs to be regularized)

- For self-adjoint, non-negative, 2nd-ord diff operator

$$L = -\Delta + V$$

Δ the Laplace-Beltrami op, V a potential depending on the classical background solution, with possibly a mass term

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- Zeta-function regularization:

$$W(\varepsilon) = S - \frac{1}{2} \int_0^\infty dt \, \frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)} \text{Tr}\, e^{-tL/\mu^2} = S - \frac{1}{2\varepsilon} \zeta(\varepsilon|L/\mu^2)$$

for the elliptic operator L the zf is def as a Mellin-like transform

$$\zeta(s|L) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{Tr}\, e^{-tL}, \qquad \zeta(s|L/\mu^2) = \mu^{2s} \zeta(s|L)$$

- Here the heat trace Tr e^{-tL} plays import role. Recall that for a 2nd-ord elliptic non-neg op L in a compact $d-$dim manifold without boundary

$$\text{Tr}\, e^{-tL} \simeq \sum_{j=0}^{\infty} A_j(L)\, t^{j-d/2}$$

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$$W(\varepsilon) = S - \frac{1}{2\varepsilon}\zeta(0|L) + \frac{\zeta(0|L)}{2}\ln \mu^2 + \frac{\zeta'(0|L)}{2} + O(\varepsilon)$$

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- More general case with log terms in heat-trace asympts. Local heat-kernel exp of Laplace type op $H = -\Delta + V(x)$. If the potential is real and non-negative, with an additional, rather mild hypothesis, the operator H is essentially self-adjoint in $C_0^\infty(\mathbb{R}^d)$

- Consider confining potentials, smooth functions giving rise to discrete spectrum [L. Parker]. Local heat-kernel expansion can be partially summed over

$$K_t(x,x) = \frac{1}{(4\pi t)^{d/2}} e^{-tV(x)} \sum_{n=0}^{\infty} b_n(x)\, t^n$$

new coeffs $b_n(x)$ easily computed, depend on the derivatives of $V(x)$

$$b_0(x) = 1, \;\; b_1(x) = 0, \;\; b_2(x) = -\frac{1}{6}\Delta V, \;\; b_3(x) = -\frac{\Delta^2 V}{60} + \frac{\nabla_k V \nabla_k V}{12}, \ldots$$

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- Here Laplace-type self-adjoint ops on non-compact manifolds. For general case of confining potent and discrete spectrum, no systematic theory [Nash]. 1-dim problems on real half-line [Voros] and Barnes zfs [Dowker]
$\longrightarrow$ Log terms appear in the abstract context of regularized products

- Under certain conditions, the regularized product assoc with an infinite sequence of non-zero complex numbers $\{\lambda_n\}$ has a related Dirichlet series $\sum_n \lambda_n^{-s}$ (the zeta function)

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- Explicit model: massive scalar field on flat spacetime $\mathbb{R} \times \mathbb{R}^3$ in external static field of confining pot which is asymptotically exponential in 2-dims. In Euclidean version, we compactify 'time' coord and third spatial coord, with periods β and l, respect.

Simple confining model

- The relevant operator [with Cognola, Zerbini]

$$L=-\frac{d^2}{d\tau^2}-\frac{d^2}{dz^2}+H_2+M^2,\quad H_2=-\Delta_2+V(r),\quad V(r)=g^2e^{\alpha^2r^2}$$

g, α dimfull parameters. Poisson's summ form and the heat-trace:

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- Since the potential is defined everywhere in $\mathbb{R}^2$, one needs a factor $e^{-tM^2/t}$

$$\zeta(s|L) \sim \frac{S}{(4\pi)^2\Gamma(s)} \sum_n \int_0^\infty dt\, t^{s+n-3} \int_{R^2} dx\, \tilde{b}_n(x)\, e^{-tV(r)}$$

$$= \sum_n \frac{\Gamma(s+n-2)}{(4\pi)^2\Gamma(s)} \int_{R^2} dx\, \tilde{b}_n(x)\, [V(r)]^{-(s+n-2)}$$

$$\tilde{b}_n = \sum_{j+k=n} \frac{(-1)^k\, b_j\, M^{2k}}{k!}, \; n \geq 2, \quad \tilde{b}_0 = 1, \quad \tilde{b}_1 = -M^2$$

$\tilde{b}_n$ have same structure as before, but now q can vanish

$$\tilde{b}_n = \sum_{pq} \tilde{C}^n_{pq} r^p a^q e^{qbr^2}, \quad 0 \leq p \leq 2(n-1), \quad 0 \leq q < n, \; n \geq 0$$

- With $\tilde{C}^n_{00} = \frac{(-1)^n M^{2n}}{n!}$. First few non-trivial b_n coefficients

$$b_2 = -\frac{2g\alpha e^{\alpha r^2}}{3}(1+\alpha r^2), \quad b_3 = -\frac{4g\alpha^2 e^{\alpha r^2}}{15}(2+4\alpha r^2+\alpha^2 r^4) + \frac{g^2\alpha^2 e^{2\alpha r^2}}{3},$$

$$b_4(x) = -\frac{\Delta^3 V}{840} + \frac{(\Delta V)^2}{72} + \frac{\nabla_i\nabla_j V \nabla_i\nabla_j V}{90} + \frac{\nabla_k V \nabla_k \Delta V}{30}, \ldots$$

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$$\zeta(s|L) = \frac{S}{16\pi\Gamma(s)} \sum_{n\geq 0;pq} \tilde{C}^n_{pq} \frac{\Gamma(s+n-2)\Gamma(1+p/2)\, a^{-(s+n-q-2)}}{b^{1+p/2}\,(s+n-q-2)^{1+p/2}}$$

Since p even, the zf has only poles of order $p/2$. The pole structure at $s=0$

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- Zeta function regularization procedure: $\longrightarrow$ needs to be modified

Proposal for extended zf regularization

- Introduce an additional spectral function depending on order of pole at the origin of the initial zf. For a pole of order N

$$\omega(s) = s^N \zeta(s|L)$$

and the definition of the regularized determinant is

$$\ln \det \frac{L}{\mu^2} = -\frac{1}{(N+1)!} \lim_{s\to 0} \frac{d^{N+1}}{ds^{N+1}} \left[\mu^{2s}\omega(s)\right]$$

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- We correspondingly define

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Relation with regularized infinite prod's

- Within the context of a general theory of regularized products [Illies 01], in cases when the zf is not holomorphic at the origin but has a first-order pole a new def of regularized product was proposed recently [Hirano 03]

$$\prod_{k=1}^{\infty} \lambda_k \equiv \exp \left[-\text{Res}_{s=0} \frac{\zeta(s)}{s^2} \right], \qquad \zeta(s) = \sum_{k=1}^{\infty} \lambda_k^{-s}$$

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- This prescription is equivalent to ours (a further consistency check, inscribes our result in a very general context)
- Shows power and flexibility of the zeta function method to deal with non-standard situations, while always fulfilling the most fundamental condition:

results obtained should reproduce measured experimental values

Collaborators

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