ICE

Zeta function regularization of quantum vacuum fluctuations in non-standard cases

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Outline of this presentation

- ΨDOs, Zeta Functions, Determinants, and Traces

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- A Generalized Zeta Function Regularization Method

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m: M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \{f \in C^\infty(\mathbb{R}^n);\ \sup_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n\}$$

- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

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ΨDOs are basic tools both in Mathematics & in Physics:

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[Fredenhagen, Brunetti, . . . R. Wald ’06]

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as

$\zeta_A(s) = \text{tr}\, A^{-s} = \sum_j \lambda_j^{-s}$, Re $s > \frac{n}{m} := s_0$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, Spec $A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(c) The definition of $\zeta_A(s)$ depends on the position of the cut L_θ

(d) The only possible singularities of $\zeta_A(s)$ are simple poles at

$$s_k = (n-k)/m, \qquad k = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

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Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr } H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} \,\text{tr } e^{-tH}$, Re $s > s_0$

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Weierstrass definition: subtract leading behavior of λ_i in i, as $i \to \infty$, until the series $\sum_{i\in I} \ln \lambda_i$ converges

$\Longrightarrow$ non-local counterterms !!

Properties

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- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum\nolimits'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum\nolimits_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum\nolimits_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \, \text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[\text{PP } \zeta_A(-k) + \psi(k+1) \, \text{Res}_{s=-k} \zeta_A(s) \right],$$

$$s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$\text{PP } \phi := \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]$$

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- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \qquad \mu_0 \geq \mu_1 \geq \cdots$$

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$$\text{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

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- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [Connes]

$$\text{Dtr}\, T = \lim_{s\to 1^+} (s-1)\zeta_{T^{-1}}(s)$$

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- Important!: it can be expressed as an integral (local form)

$$\text{res } A = \int_{S^*M} \text{tr } a_{-n}(x, \xi) \, d\xi$$

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- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

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- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x,\xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1}=0$:

  $\text{Res}_{s=s_k}\zeta_A(s) = \frac{1}{m}\,\text{res}\; A^{-s_k} = \frac{1}{m}\int_{S^*M} \text{tr}\; a_{-n}^{-s_k}(x,\xi)\, d^{n-1}\xi$

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- Proof. The homog component of degree $-n$ of the corresp power of the principal symbol of A is obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$:

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial\xi^{-1}}\right)^k \left[\xi^{n-k}a^{(k-n)/m}(x,\xi)\right]\Big|_{\xi^{-1}=0}\,\xi^{-n}$$

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

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- The multiplicative (or noncommutative) anomaly (defect) is defined as

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- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2\,\text{ord}\,A\,\text{ord}\,B\,(\text{ord}\,A + \text{ord}\,B)}$$

where $\quad \sigma(A,B) = A^{\text{ord}\,B} B^{-\text{ord}\,A}$

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi] \; \exp\left\{ - \int d^D x \left[\Phi^\dagger(x)(\quad)\Phi(x) + \cdots \right] \right\}$$

Gaussian integration: $\longrightarrow \quad \det(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

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- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$

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- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$

- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

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$$\sum_{\lambda=1}^{|D|} \left(\frac{D}{\lambda}\right) \log \Gamma\left(\frac{\lambda}{D}\right) = h \log|D| - \frac{h}{3}\log(2\pi) - \sum_{(a,b,c)} \log a$$
$$+\frac{2}{3}\sum_{(a,b,c)} \log\left[\theta_1'(0|\alpha)\theta_1'(0|\beta)\right]$$

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- Eta evaluations Dedekind eta function for Im $(\tau) > 0$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q := e^{2\pi i\tau}$$

It is a 24-th root of the discriminant func $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \,|\, a, b \in \mathbb{Z}\}$

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}$$

Properties & Recent Results

⟹ The C-S formula gives the value of a product of eta functions

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Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau\in\mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \qquad z,t\in\mathbb{C},\ \text{Re}\, t>0$$

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[Gelbart + Miller, BAMS '03, Iwaniec, Morgan, ICM '06]

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- Truncated sums

$\longrightarrow$ asymptotic series

Extended CS Formulas (ECS)

- Consider the zeta function (Re $s > p/2, A > 0,$ Re $q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}$$

prime: point $\vec{n} = \vec{0}$ to be excluded from the sum

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- Case $q \neq 0$ (Re $q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)}$$

$$\times \sum_{\vec{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c})\left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

[ECS1]

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[ECS1]

- Pole: $s = p/2$ Residue:

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive]

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- Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s}\, \Gamma\left(s-\frac{j}{2}\right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right)\right]$$

[ECS3d]

QFT in s-t with non-comm toroidal part

- $D-$dim non-commut manifold: $M = \mathbb{R}^{1,d} \bigotimes \mathbb{T}^p_\theta, \quad D = d + p + 1$

  $\mathbb{T}^p_\theta$ a $p-$dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$

  σ_{jk} a real, nonsingular, antisymmetric matrix of ± 1 entries

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- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s) = \frac{V\,\Gamma(s-(d+1)/2)}{(4\pi)^{(d+1)/2}\,\Gamma(s)} \sum_{\vec{n}\in\mathbb{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s}\left[1+\Lambda\theta^{2-2\alpha}Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s}$$

$\alpha = 2$ bos, $\alpha = 3$ ferm, $V = \text{Vol}\,(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j = a_j^{-1/2}$ compactific radii

- After some calculations,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} \left(-\Lambda\theta^{2-2\alpha}\right)^l \zeta_{Q,\vec{0},0}(s+\alpha l - \frac{d+1}{2})$$

for all radii equal to R, with $I(\vec{n}) = \sum_{j=1}^{p} n_j^2$,

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where we use the notation $\zeta_E(s) := \zeta_{I,\vec{0},0}(s)$

e.g., the Epstein zeta function for the standard quadratic form

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- Classify the different possible cases according to the values of d and $D = d+p+1$. We obtain, at $s=0$:

$$\text{For } d = 2k \quad \begin{cases} \text{if } D \neq \overline{2\alpha}^{\,\cdot} & \Longrightarrow \quad \zeta_\alpha(0) = 0 \\ \text{if } D = \overline{2\alpha}^{\,\cdot} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \end{cases}$$

$$\text{For } d = 2k-1 \quad \begin{cases} \text{if } D \neq \overline{2\alpha}^{\,\cdot} \left\{ \begin{array}{ll} \text{finite,} & \text{for } l \leq k \\ 0, & \text{for } l > k \end{array} \right\} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \\ \\ \text{if } D = 2\alpha l \left\{ \begin{array}{ll} \text{pole,} & \text{for } l \leq k \\ \text{finite,} & \text{for } l > k \end{array} \right\} & \Longrightarrow \quad \zeta_\alpha(0) = \text{pole} \end{cases}$$

Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\overline{2\alpha}^{\,\cdot}$ means multiple of 2α)

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Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means multiple of 2α)

$\Longrightarrow$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$,
& specific pole structure

$$\zeta_\alpha(s) = \frac{2^{s-d}\,V}{(2\pi)^{(d+1)/2}\Gamma(s)}\sum_{l=0}^{\infty}\frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)}(-2^\alpha\Lambda\theta^{2-2\alpha})^l\sum_{j=0}^{p-1}(\det A_j)^{-\frac{1}{2}}$$

$$\times\Big[\pi^{j/2}a_{p-j}^{-s-\alpha l+(d+j+1)/2}\Gamma(s+\alpha l-(d+j+1)/2)\zeta_R(2s+2\alpha l-d-j-1)$$

$$+\,4\pi^{s+\alpha l-(d+1)/2}a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4}\sum_{n=1}^{\infty}\sum_{\vec{m}_j\in\mathbb{Z}^j}{}'\,n^{(d+j+1)/2-s-\alpha l}$$

$$\times\left(\vec{m}_j^t A_j^{-1}\vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4}K_{(d+j+1)/2-s-\alpha l}\left(2\pi n\sqrt{a_{p-j}\vec{m}_j^t A_j^{-1}\vec{m}_j}\right)\Big]$$

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p \ D	even	odd
odd	(1a) pole / finite ($l \geq l_1$)	(2a) pole / pole
even	(1b) double pole / pole ($l \geq l_1, l_2$)	(2b) pole / double pole ($l \geq l_2$)

General pole structure of $\zeta_\alpha(s)$, for the possible values of D and p being odd or even. Magenta, type of behavior corresponding to lower values of l; behavior in blue corresponds to larger values of l

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- For a physically meaningful evaluation of the functional determinant, we propose a generalized zeta-f reg procedure
- One-loop approx in QFT: Euclidean 1ℓ effective action is sum of classical action and a functional determinant of an elliptic diff op: the fluctuation operator (needs to be regularized)

- For self-adjoint, non-negative, 2nd-ord diff operator

$$L = -\Delta + V$$

Δ the Laplace-Beltrami op, V a potential depending on the classical background solution, and possibly a mass term

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- Zeta-function regularization:

$$W(\varepsilon) = S - \frac{1}{2} \int_0^\infty dt \, \frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)} \text{Tr}\, e^{-tL/\mu^2} = S - \frac{1}{2\varepsilon} \zeta(\varepsilon | L/\mu^2)$$

for the elliptic operator L the zf is def as a Mellin-like transform

$$\zeta(s|L) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{Tr}\, e^{-tL}, \qquad \zeta(s|L/\mu^2) = \mu^{2s} \zeta(s|L)$$

- Here the heat trace $\mathrm{Tr}\, e^{-tL}$ plays import role. Recall that for a 2nd-ord elliptic non-neg op L in a compact $d-$dim manifold without boundary

$$\mathrm{Tr}\, e^{-tL} \simeq \sum_{j=0}^{\infty} A_j(L)\, t^{j-d/2}$$

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$$W(\varepsilon) = S - \frac{1}{2\varepsilon}\zeta(0|L) + \frac{\zeta(0|L)}{2}\ln\mu^2 + \frac{\zeta'(0|L)}{2} + O(\varepsilon)$$

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- More general case with log terms in heat-trace asympts. Local heat-kernel exp of Laplace type op $H = -\Delta + V(x)$. If potential real and non-negative, with an additional, rather mild hypothesis, the operator H is essentially self-adjoint in $C_0^\infty(\mathbb{R}^d)$.

- Consider confining potentials, smooth functions giving rise to discrete spectrum [L. Parker]. Local heat-kernel expansion can be partially summed over

$$K_t(x,x) = \frac{1}{(4\pi t)^{d/2}} e^{-tV(x)} \sum_{n=0}^{\infty} b_n(x)\, t^n$$

new coeffs $b_n(x)$ easily computed, depend on the derivatives of $V(x)$

$$b_0(x) = 1, \;\; b_1(x) = 0, \;\; b_2(x) = -\frac{1}{6}\Delta V, \;\; b_3(x) = -\frac{\Delta^2 V}{60} + \frac{\nabla_k V \nabla_k V}{12}, \ldots$$

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- Here Laplace-type self-adjoint ops on non-compact manifolds. For general case of confining potent and discrete spectrum, no systematic theory [Nash]. 1-dim problems on real half-line [Voros] and Barnes zfs [Dowker]
$\longrightarrow$ Log terms appear in the abstract context of regularized products

- Under certain conditions, the regularized product assoc with an infinite sequence of non-zero complex numbers $\{\lambda_n\}$ has a related Dirichlet series $\sum_n \lambda_n^{-s}$ (the zeta function)

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- Explicit model: massive scalar field on flat spacetime $\mathbb{R} \times \mathbb{R}^3$ in external static field of confining pot which is asymptotically exponential in 2-dims. In Euclidean version, we compactify 'time' coord and third spatial coord, with periods β and l, respect.

Simple confining model

- The relevant operator

$$L = -\frac{d^2}{d^2\tau} - \frac{d^2}{dz^2} + H_2 + M^2, \quad H_2 = -\Delta_2 + V(r), \quad V(r) = g^2 e^{\alpha^2 r^2}$$

g, α dim-full parameters. Poisson's summ formula and the heat-trace:

$\mathrm{Tr}\, e^{-tL} = \frac{S\, e^{-tM^2}}{4\pi t} \mathrm{Tr}\, e^{-tH_2} + \cdots$, $S = \beta\, l$, dots are exp small terms in t

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- Since the potential is defined everywhere in $\mathbb{R}^2$, one needs a factor $e^{-tM^2/t}$

$$\zeta(s|L) \sim \frac{S}{(4\pi)^2\Gamma(s)} \sum_n \int_0^\infty dt\, t^{s+n-3} \int_{R^2} dx\, \tilde{b}_n(x)\, e^{-tV(r)}$$

$$= \sum_n \frac{\Gamma(s+n-2)}{(4\pi)^2\Gamma(s)} \int_{R^2} dx\, \tilde{b}_n(x)\, [V(r)]^{-(s+n-2)}$$

$$\tilde{b}_n = \sum_{j+k=n} \frac{(-1)^k\, b_j\, M^{2k}}{k!}, \; n \geq 2, \quad \tilde{b}_0 = 1, \quad \tilde{b}_1 = -M^2$$

the $\tilde{b}_n$ have same structure as before, but now q can vanish

$$\tilde{b}_n = \sum_{pq} \tilde{C}^n_{pq}\, r^p a^q\, e^{qbr^2}, \quad 0 \leq p \leq 2(n-1), \quad 0 \leq q < n, \; n \geq 0$$

- With $\tilde{C}_{00}^{n} = \frac{(-1)^n M^{2n}}{n!}$. First few non-trivial b_n coefficients

$$b_2 = -\frac{2g\alpha e^{\alpha r^2}}{3}(1+\alpha r^2), \quad b_3 = -\frac{4g\alpha^2 e^{\alpha r^2}}{15}(2+4\alpha r^2+\alpha^2 r^4) + \frac{g^2\alpha^2 e^{2\alpha r^2}}{3},$$
$$b_4(x) = -\frac{\Delta^3 V}{840} + \frac{(\Delta V)^2}{72} + \frac{\nabla_i\nabla_j V \nabla_i \nabla_j V}{90} + \frac{\nabla_k V \nabla_k \Delta V}{30}, \ldots$$

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$$\zeta(s|L) = \frac{S}{16\pi\Gamma(s)} \sum_{n\geq 0;pq} \tilde{C}^n_{pq} \frac{\Gamma(s+n-2)\Gamma(1+p/2)\, a^{-(s+n-q-2)}}{b^{1+p/2}\,(s+n-q-2)^{1+p/2}}$$

Since p even, the zf has only poles of order $p/2$. The pole structure at $s=0$

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- The zeta function regularization procedure needs to be modified

Proposal for extended zf regularization

- Introduce an additional spectral function depending on order of pole at the origin of the initial zf. For a pole of order N

$$\omega(s) = s^N \zeta(s|L)$$

and the definition of the regularized determinant is

$$\ln \det \frac{L}{\mu^2} = -\frac{1}{(N+1)!} \lim_{s\to 0} \frac{d^{N+1}}{ds^{N+1}} \left[\mu^{2s}\omega(s)\right]$$

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- We correspondingly define

$$\ln \det \frac{L}{\mu^2} = -\frac{1}{3!} \lim_{s \to 0} \frac{d^3}{ds^3} \left[\mu^{2s}\omega(s)\right]$$

Relation with regularized infinite prod's

- Within the context of a general theory of regularized products [Illies], in the case when the zf is not holomorphic at the origin, but has a first-order pole, a new def of regularized product has been proposed recently [Hirano]

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- It shows the power and flexibility of the zf method to deal with non-standard situations, without ever loosing contact with the most fundamental issue:

$\longrightarrow$ *the results obtained match measured experimental values*