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Quadratic zeta functions and the Chowla-Selberg formula

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- The Chowla-Selberg Expansion Formula (CS)
- Extended Chowla-Selberg Expressions (ECS)

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m: M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \{f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n\}$$

- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

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ΨDOs are basic tools both in Mathematics & in Physics:

1. Proof of uniqueness of Cauchy problem (Calderón-Zygmund)

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian (Seeley, Gilkey, ...)

4. Constitute nowadays the basic starting point of any rigorous formulation of QFT field theory through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

(Fredenhagen, Brunetti, ...)

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as $\zeta_A(s) = \text{tr}\ A^{-s} = \sum_j \lambda_j^{-s}$, Re $s > \frac{n}{m} \equiv s_0$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A ($a_m(x, \xi)$) admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg}\, \lambda = \theta, \theta_1 < \theta < \theta_2\}$, Spec $A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(c) The definition of $\zeta_A(s)$ depends on the position of the cut L_θ

(d) The only possible singularities of $\zeta_A(s)$ are simple poles at

$$s_k = (n-k)/m, \qquad k = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

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Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr}\, H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1}\, \text{tr}\, e^{-tH}, \quad Re\, s > s_0$

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Weierstrass definition: subtract leading behavior of λ_i in i, as $i \to \infty$, until the series $\sum_{i\in I} \ln \lambda_i$ converges

$\Longrightarrow$ non-local counterterms !!

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- Asymptotic expansion for the heat kernel:

$$\text{tr}\, e^{-tA} = \sum'_{\lambda \in \text{Spec}\, A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[PP \zeta_A(-k) + \psi(k+1) \text{Res}_{s=-k} \zeta_A(s) \right],$$

$$s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N} \backslash \{0\}$$

$$PP\phi = \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p}\, \phi(s)}{s-p} \right]$$

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- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

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$$\text{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

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- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ (Connes)

$$\text{Dtr } T = \lim_{s\to 1^+} (s-1) \zeta_{T^{-1}}(s)$$

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  $$\text{res}\, A = \int_{S^*M} \text{tr}\, a_n(x, \xi)\, d\xi$$

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- The Wodzicki res makes sense for ΨDOs of arbitrary order. Even if symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

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- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x,\xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1}=0$:

  $$\text{Res}_{s=s_k}\zeta_A(s) = \frac{1}{m}\,\text{res}\ A^{-s_k} = \frac{1}{m}\int_{S^*M} \text{tr}\ a_{-n}^{-s_k}(x,\xi)\,d^{n-1}\xi$$

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- Proof. Homog component of degree $-n$ of the corresp power of the principal symbol of A are obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial\xi^{-1}}\right)^k \left[\xi^{n-k}a^{(k-n)/m}(x,\xi)\right]\Big|_{\xi^{-1}=0}\ \xi^{-n}$$

Multiplicative Anomaly (or Defect)

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

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- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2 \,\text{ord}\, A \,\text{ord}\, B \,(\text{ord}\, A + \text{ord}\, B)}$$

where $\quad \sigma(A,B) = A^{\text{ord}\, B} B^{-\text{ord}\, A}$

Consequences of the m.a.

- In the path integral formulation

$$\int [d\Phi]\ \exp\left\{-\int d^Dx \left[\Phi^\dagger(x)(\quad)\Phi(x)+\cdots\right]\right\}$$

Gaussian integration: $\longrightarrow \quad \det\,(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

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- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$

Consequences of the m.a.

- In the path integral formulation

$$\int [d\Phi]\ \exp\left\{-\int d^Dx\left[\Phi^\dagger(x)(\quad)\Phi(x)+\cdots\right]\right\}$$

Gaussian integration: $\longrightarrow \quad \det(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \longrightarrow \begin{pmatrix} A & \\ & B \end{pmatrix}$$

$\det(AB)$ or $\det A \cdot \det B$?

- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$
- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

The Chowla-Selberg Formula (CS)

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ON EPSTEIN'S ZETA FUNCTION (I)

BY S. CHOWLA AND A. SELBERG

INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.

Communicated by H. Weyl, May 18, 1949

1. This paper contains a short account of results whose detailed proofs will be published later.

We define the function $Z(s)$ by

$$Z(s) = \sum{}'(am^2 + bmn + cn^2)^{-s} \tag{1}$$

where $s = \sigma + it$ (σ and t, real), $\sigma > 1$, and the summation is for all integers m, n (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further a and c are positive numbers while b is real and subject to $4ac - b^2 = \Delta > 0$.

It is well known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole s-plane, and satisfies a functional equation similar to the one satisfied by the Riemann Zeta Function. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

Deuring (*Math. Ztschr.*, **37**, 403–413 (1933)) obtained an important formula for $Z(s)$. Deuring's work led Heilbronn (*Quart. J. Maths., Oxford*, **5**, 150 (1934)) to the proof of the following famous conjecture of Gauss on the class-number of binary quadratic forms with a negative fundamental discriminant: let $h(-\Delta)$ denote the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, then

$$h(-\Delta) \to \infty \quad \text{as} \quad \Delta \to \infty \tag{2}$$

Again using the ideas of Heilbronn and Deuring, Siegel proved that

$$h(-\Delta) > \Delta^{1/2 - \epsilon}\ [\Delta > \Delta_0(\epsilon)] \tag{3}$$

which is a great advance on (2).

Our starting point is the formula:

$$Z(s) = 2\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\,\Delta^{s-1/2}}\zeta(2s-1)\Gamma(s - 1/2) + Q(s) \tag{4}$$

where

$$Q(s) = \frac{\pi^s \cdot 2^{s+3/2}}{a^{1/2}\Gamma(s)\,\Delta^{s/2 - 1/4}} \sum_{n=1}^{\infty} n^{s-1/2}\,\sigma_{1-2s}(n) \cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \phi^{s-3/2} \exp\left\{-\frac{\pi n \Delta^{1/2}}{2a}(\phi + \phi^{-1})\right\} d\phi \tag{4}$$

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W. Brödel, M. Deuring, A. Grothendieck, P. R. Halmos, O. Haupt,
F. Hirzebruch, E. Hopf, M. Kneser, G. Köthe, K. Prachar, H. Reichardt,
P. Roquette, W. Schmeidler, L. Schmetterer, E. Stiefel

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On Epstein's Zeta-function

By *Atle Selberg* at Princeton (N. J.), and *S. Chowla* at State College (Pa.)

Introduction

This paper was written in the Spring of 1949, and a resumé appeared in the note: On Epstein's zeta Function (I), Proceedings of the National Academy of Sciences (U. S. A.), **35** (1949), 371—374.

Meanwhile, the following papers which have reference to the Proceedings paper, came to our attention:

1. *J. B. Rosser*, Real roots of real Dirichlet L-series, Jour. Research National Bureau of Standards, **45** (1950), 505—514.

2. *E. A. Anferteva*, On an identity of Chowla and Selberg (Russian), Izvestija Vysšik Učebnyh Zavadeniĭ Mathematika (Kazan), No. 3 (10) (1959), 13—21.

3. *P. T. Bateman* and *E. Grosswald*, On Epstein's zeta Function, Acta Arithmetica, **9** (1964), 365—373.

4. *K. Ramachandra*, Some applications of Kronecker's limit formulas, Annals of Mathematics **80** (1964), 104—148.

§ 1.

We define the function $Z(s)$ by

$$Z(s) = \Sigma' (am^2 + bmn + cn^2)^{-s} \tag{1}$$

where $s = \sigma + it$ (σ and t, real), $\sigma > 1$, and the summation is for all integers m, n (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further a and c are positive numbers while b is real and subject to $4ac - b^2 = \Delta > 0$.

It is well-known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole s-plane. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

In 1933, Deuring obtained an important formula for $Z(s)$. Deuring's work led Heilbronn to his proof of a famous conjecture of Gauss on the class number of binary quadratic forms with a negative fundamental discriminant. If $h(-\Delta)$ is the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, Gauss conjectured that

$$h(-\Delta) \to \infty \text{ as } \Delta \to \infty. \tag{2}$$

Transforming this we get

$$\sum_{j=1}^{h} \log \Delta\left(\frac{b_j + i\sqrt{|d|}}{2a_j}\right) = 6\left\{h\gamma + \log \frac{\prod_{j=1}^{h} a_j}{|d|^h}\right\} - \frac{3w}{\pi}\sqrt{|d|}\, L'_d(1).$$

Inserting here the value (obtained like (58))

$$L'_d(1) = -\frac{\pi}{\sqrt{|d|}} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \log \Gamma\left(\frac{m}{|d|}\right) + \frac{2h\pi(\gamma + \log 2\pi)}{w\sqrt{|d|}}$$

one gets, writing $\tau_j = \dfrac{b_j + i\sqrt{|d|}}{2a_j}$,

$$\prod_{j=1}^{h} \Delta(\tau_j) = \frac{\prod_{j=1}^{h} a_j^6}{(2\pi\,|d|)^{6h}} \left\{ \prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)} \right\}^{3w} \tag{2}$$

Now let $\tau = i\dfrac{K'}{K}$ be a number from the field $k(\sqrt{d})$, then from Lemma 3 we get

$$\frac{\Delta(\tau_j)}{\Delta(\tau)} = \lambda_j,$$

where λ_j are algebraic numbers. Thus (2) gives

$$\Delta(\tau) = \frac{\lambda'}{\pi^6} \left\{ \prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)} \right\}^{\frac{3w}{h}}, \tag{3}$$

where λ' is an algebraic number. Finally we have from (48)

$$\Delta(\tau) = \left(\frac{2K}{\pi}\right)^{12} \cdot 2^{-8}(kk')^4 = \lambda''\left(\frac{K}{\pi}\right)^{12},$$

where λ'' is an algebraic number. This gives, when inserted in (3)

$$K = \lambda'''\sqrt{\pi} \left\{ \prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)} \right\}^{\frac{w}{4h}}, \tag{4}$$

which is the desired expression for K in finite terms.

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History

- Lerch (1897):

$$\sum_{\lambda=1}^{|D|} \left(\frac{D}{\lambda}\right) \log\Gamma\left(\frac{\lambda}{D}\right) = h\log|D| - \frac{h}{3}\log(2\pi) - \sum_{(a,b,c)} \log a + \frac{2}{3}\sum_{(a,b,c)} \log\left[\theta_1'(0|\alpha)\theta_1'(0|\beta)\right]$$

D discriminant, $\theta_1' \sim \eta^3$

h class number of binary quadratic forms (a,b,c)

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- Eta evaluations Dedekind eta function for Im $(\tau) > 0$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q := e^{2\pi i\tau}$$

It is a 24-th root of the discriminant func $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \,|\, a, b \in \mathbb{Z}\}$

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}$$

Properties & Recent Results

⟹ The C-S formula gives the value of a product of eta functions

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- W.B. Hart, PhD Thesis, 2004 (Macquarie U., Sidney)

Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau\in\mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \qquad z,t\in\mathbb{C},\ Re\, t>0$$

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- Higher dimensions: Poisson summ formula (Riemann)

$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p} \tilde{f}(\vec{m})$$

$\tilde{f}$ Fourier transform, Gelbart + Miller, BAMS, Oct. '03

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- Truncated sums

$\longrightarrow$ asymptotic series

Extended CS Formulas (ECS)

- Consider the zeta function ($Re\, s > p/2, A > 0, Re\, q > 0$):

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in \mathbf{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} = \sum_{\vec{n}\in \mathbf{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}$$

prime: point $\vec{n} = \vec{0}$ to be excluded from the sum
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- Case $q \neq 0$ $(Re\, q > 0)$

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)}$$
$$\times \sum_{\vec{m}\in \mathbf{Z}^p_{1/2}}{}' \cos(2\pi \vec{m}\cdot\vec{c}) \left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\, \vec{m}^T A^{-1}\vec{m}}\right)$$

[ECS1]

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$$\times \sum_{\vec{m}\in \mathbf{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c}) \left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

[ECS1]

- Pole: $s = p/2$ Residue:

$$\text{Res}_{s=p/2}\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A)^{-1/2}$$

- Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one can select those vectors in $\mathbb{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive]

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- Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s} \, \Gamma\left(s - \frac{j}{2}\right) \zeta_R(2s-j) + \right.$$

$$\left. 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right) \right]$$

[ECS3d]