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On zeta regularization and some of its uses in cosmology

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- Pseudodifferential Operators [ΨDO]

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- Conclusions & Outlook

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m: M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x,\xi) \right| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function ($\mathcal{S}$ Schwartz's space)
$$f \in \mathcal{S} = \{f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n\}$$
- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $\quad a_k(x,\xi) = b_k(x)\,\xi^k$

– $a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

– An elliptic ΨDO is one with an elliptic symbol

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ΨDOs are basic tools both in Mathematics & Physics

1. Proof of uniqueness of Cauchy problem (Calderón-Zygmund)

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian (Seeley, Gilkey, ...)

4. Constitute nowadays the basic starting point of any rigorous formulation of QFT field theory through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[Fredenhagen, Brunetti, ...]

Existence of ζ_A for A a ΨDO

- A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}$
- A acts on the space of smooth sections of
- E, n-dim vector bundle over
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(a) The zeta function is defined as $\zeta_A(s) = \text{tr}\, A^{-s} = \sum_j \lambda_j^{-s}$, $\text{Re}\, s > \frac{n}{m} \equiv s_0$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C} \,|\, \text{Arg}\, \lambda = \theta, \theta_1 < \theta < \theta_2\}$, Spec $A \cap L_\theta = \emptyset$

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(d) The only possible singularities of $\zeta_A(s)$ are simple poles at

$$s_k = (n-k)/m, \qquad k = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

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Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \; Re\, s > 1$ (& analytic cont)

$\Longrightarrow$ Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr}\, H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1}\, \text{tr}\, e^{-tH}, \quad Re\, s > D/2$

Derivative: $\zeta_H'(0) = -\sum_{i\in I} \ln \lambda_i$

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Weierstrass definition: subtract leading behavior of λ_i in i, as $i \to \infty$, until the series $\sum_{i\in I} \ln \lambda_i$ converges

$\Longrightarrow$ non-local counterterms !!

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- Asymptotic expansion for the heat kernel

$$\text{tr}\, e^{-tA} = \sum\nolimits'_{\lambda \in \text{Spec}\, A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum\nolimits_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum\nolimits_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j)\, \text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[\text{PP}\, \zeta_A(-k) + \psi(k+1)\, \text{Res}_{s=-k} \zeta_A(s)\right], \qquad s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$\text{PP}\, \phi = \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p}\, \phi(s)}{s-p}\right]$$

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- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) \equiv \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \qquad \mu_0 \geq \mu_1 \geq \cdots$$

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- Definition of the Dixmier trace of T:

$$\text{Dtr } T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

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- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ (Connes)

$$\text{Dtr } T = \lim_{s\to 1^+} (s-1)\zeta_{T^{-1}}(s)$$

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$$\text{res}\, A = \int_{S^*M} \text{tr}\, a_n(x,\xi)\, d\xi$$

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- The Wodzicki res makes sense for ΨDOs of arbitrary order. Even if symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

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- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x,\xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1}=0$, then

$$\text{Res}_{s=s_k}\zeta_A(s) = \frac{1}{m}\,\text{res}\ A^{-s_k} = \frac{1}{m}\int_{S^*M} \text{tr}\ a_{-n}^{-s_k}(x,\xi)\, d^{n-1}\xi$$

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- Proof. The homog component of degree $-n$ of the corresp power of the principal symbol of A is obtained as the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$:

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial \xi^{-1}}\right)^k \left[\xi^{n-k} a^{(k-n)/m}(x,\xi)\right]\Bigg|_{\xi^{-1}=0} \xi^{-n}$$

Multiplicative Anomaly (or Defect)

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- The multiplicative (or noncommutative) anomaly (or defect) is defined as

$$\delta(A, B) = \ln \left[\frac{\det_\zeta(AB)}{\det_\zeta A \ \det_\zeta B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

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- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det{}_\zeta(AB) \neq \det{}_\zeta A \, \det{}_\zeta B$$

- The multiplicative (or noncommutative) anomaly (or defect) is defined as

$$\delta(A,B) = \ln\left[\frac{\det_\zeta(AB)}{\det_\zeta A \ \det_\zeta B}\right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2 \ \text{ord} \, A \ \text{ord} \, B \ (\text{ord} \, A + \text{ord} \, B)}$$

where $\quad \sigma(A,B) = A^{\text{ord}\, B} B^{-\text{ord}\, A}$

Consequences of the m.a.

- In the path integral formulation

$$\int [d\Phi] \;\exp\left\{ - \int d^D x \left[\Phi^\dagger(x) (\quad) \Phi(x) + \cdots \right] \right\}$$

Gaussian integration: $\longrightarrow \quad \det\,(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

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The Chowla-Selberg Formula (CS)

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$$\sum_{\lambda=1}^{|D|} \left(\frac{D}{\lambda}\right) \log\Gamma\left(\frac{\lambda}{D}\right) = h\log|D| - \frac{h}{3}\log(2\pi) - \sum_{(a,b,c)} \log a + \frac{2}{3}\sum_{(a,b,c)} \log\left[\theta_1'(0|\alpha)\theta_1'(0|\beta)\right]$$

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- Eta evaluations: Dedekind eta function for Im $(\tau) > 0$

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n), \qquad q := e^{2\pi i\tau}$$

It is a 24-th root of the discriminant func $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \,|\, a, b \in \mathbb{Z}\}$

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

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$\Longrightarrow$ The C-S formula gives the value of a product of eta functions

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Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau\in\mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right),$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \qquad z,t\in\mathbb{C},\ \text{Re}\, t>0$$

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$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p} \tilde{f}(\vec{m})$$

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- Truncated sums

$\longrightarrow$ asymptotic series

Extended CS Formulas (ECS)

- Consider the zeta function $(Re\, s > p/2, A > 0, Re\, q > 0)$

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}$$

prime means point $\vec{n}=\vec{0}$ to be excluded from sum

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[ECS1]

- Pole: $s = p/2$ Residue:

$$\text{Res}_{s=p/2}\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}(\det A)^{-1/2}$$

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[ECS3d]

Vacuum energy density and the CC

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  energy ALWAYS gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu}$

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- Idea: zero point fluctuations do contribute to the cosmological constant

Cosmo-Topological Casimir Effect (I)

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- What we do consider —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:
  $\Longrightarrow$ kind of cosmological Casimir effects

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- Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until very recently), we will be left with this incremental value coming from the topology or BCs
- We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in a number of reasonable models involving:
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A. Simple model: large & small dim's

- Space-time:

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- M mass of the field arbitrarily small (a tiny mass for the field can never be excluded); see L. Parker & A. Raval, PRL86 749 (2001); PRD62 083503 (2000)

- For d–open, (p, q)–toroidal universe:

$$\rho_\phi = \frac{\pi^{-d/2}}{2^d \Gamma(d/2) \prod_{j=1}^{p} a_j \prod_{h=1}^{q} b_h} \int_0^\infty dk\, k^{d-1} \sum_{\mathbf{n}_p=-\infty}^{\infty} \sum_{\mathbf{m}_q=-\infty}^{\infty} \left[\sum_{j=1}^{p} \left(\frac{2\pi n_j}{a_j} \right)^2 + \sum_{h=1}^{q} \left(\frac{2\pi m_h}{b_h} \right)^2 + \mathbf{k}_d^2 + M^2 \right]^{1/2}$$

$$\simeq \frac{1}{a^p b^q} \sum_{\mathbf{n}_p, \mathbf{m}_q=-\infty}^{\infty} \left(\frac{1}{a^2} \sum_{j=1}^{p} n_j^2 + \frac{1}{b^2} \sum_{h=1}^{q} m_h^2 + M^2 \right)^{\frac{d+1}{2}+1}$$

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$$\rho_\phi = \frac{\pi^{-d/2}}{2^d\Gamma(d/2)\prod_{j=1}^p a_j\, b^q}\int_0^\infty dk\, k^{d-1}\sum_{\mathbf{n}_p=-\infty}^{\infty}\sum_{l=1}^{\infty}P_{q-1}(l)$$

$$\left[\sum_{j=1}^p\left(\frac{2\pi n_j}{a_j}\right)^2+\frac{Q_2(l)}{b^2}+\mathbf{k}_d^2+M^2\right]^{1/2}$$

$[P_{q-1}(l)$ poly in l deg $q-1]$

$$\simeq \frac{1}{a^p b^q}\sum_{\mathbf{n}_p=-\infty}^{\infty}\sum_{l=1}^{\infty}P_{q-1}(l)\left(\frac{4\pi^2}{a^2}\sum_{j=1}^p n_j^2+\frac{l(l+q)}{b^2}+\frac{M^2}{4\pi^2}\right)^{\frac{d+1}{2}+1}$$

- Regularize: zeta function

$$\zeta(s) = \frac{1}{2}\sum_i \lambda_i^{-s} = \frac{1}{2}\sum_{\mathbf{k}} \left(\frac{k^2 + M^2}{\mu^2}\right)^{-s/2} \Longrightarrow \rho_\phi = \zeta(-1)$$

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- For the zeta function (Re $s > p/2$):

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}(\vec{n}+\vec{c})^T A(\vec{n}+\vec{c}) + q\right]^{-s}$$
$$\equiv \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q(\vec{n}+\vec{c}) + q\right]^{-s}$$

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$$\sum_{\vec{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c})\,\left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m}\in\mathbb{Z}^p$ are summed over. That is, if we take an $\vec{m}\in\mathbb{Z}^p$ we must then exclude $-\vec{m}$ (as simple criterion one can, for instance, select those vectors in $\mathbb{Z}^p\backslash\{\vec{0}\}$ whose first non-zero component is positive).

Calcul. of vacuum energy density

Analytic continuation of the zeta function:

$$\zeta(s) = \frac{2\pi^{s/2+1}}{a^{p-(s+1)/2} b^{q-(s-1)/2} \Gamma(s/2)} \sum_{\mathbf{m}_q=-\infty}^{\infty} \sum_{h=0}^{p} \binom{p}{h} 2^h \sum_{\mathbf{n}_h=1}^{\infty} \left(\frac{\sum_{j=1}^{h} n_j^2}{\sum_{k=1}^{q} m_k^2 + M^2} \right)^{\frac{s-1}{4}}$$

$$\times K_{(s-1)/2} \left[\frac{2\pi a}{b} \left(\sum_{j=1}^{h} n_j^2 \right)^{1/2} \left(\sum_{k=1}^{q} m_k^2 + M^2 \right)^{1/2} \right]$$

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Yields the vacuum energy density:

$$\rho_\phi = -\frac{1}{a^p b^{q+1}} \sum_{h=0}^{p} \binom{p}{h} 2^h \sum_{\mathbf{n}_h=1}^{\infty} \sum_{\mathbf{m}_q=-\infty}^{\infty} \sqrt{\frac{\sum_{k=1}^{q} m_k^2 + M^2}{\sum_{j=1}^{h} n_j^2}}$$

$$\times K_1 \left[\frac{2\pi a}{b} \left(\sum_{j=1}^{h} n_j^2 \right)^{1/2} \left(\sum_{k=1}^{q} m_k^2 + M^2 \right)^{1/2} \right]$$

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Inserting the $\hbar$ and c factors

$$\rho_\phi = -\frac{\hbar c}{2\pi a^{p+1} b^q}\left[1 + \sum_{h=0}^{p}\binom{p}{h} 2^h \alpha\right] + \mathcal{O}\left[q K_1\left(\frac{2\pi a}{b}\right)\right]$$

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$\Longrightarrow$ Sign may change with BC (e.g., Dirichlet): a problem

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ρ_ϕ	$p=0$	$p=1$	$p=2$	$p=3$
$b = l_P$	10^{-13}	10^{-6}	1	10^{5}
$b = 10 l_P$	10^{-14}	$[10^{-8}]$	10^{-3}	10
$b = 10^2 l_P$	10^{-15}	(10^{-10})	10^{-6}	10^{-3}
$b = 10^3 l_P$	10^{-16}	10^{-12}	$[10^{-9}]$	(10^{-7})
$b = 10^4 l_P$	10^{-17}	10^{-14}	10^{-12}	10^{-11}
$b = 10^5 l_P$	10^{-18}	10^{-16}	10^{-15}	10^{-15}

Table 2: Vacuum energy density in units of erg/cm^3, for p large compactified dimensions a, and $q = p + 1$ small compactified dimensions b, $p = 0, \ldots, 3$, for different values of b, proportional to the Planck length l_P

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$\Longrightarrow$ To examine $\rightarrow$ couplings in GR

$\rightarrow$ alternative theories

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Elizalde, Nojiri, Odintsov, PRD70 (2004) 043539

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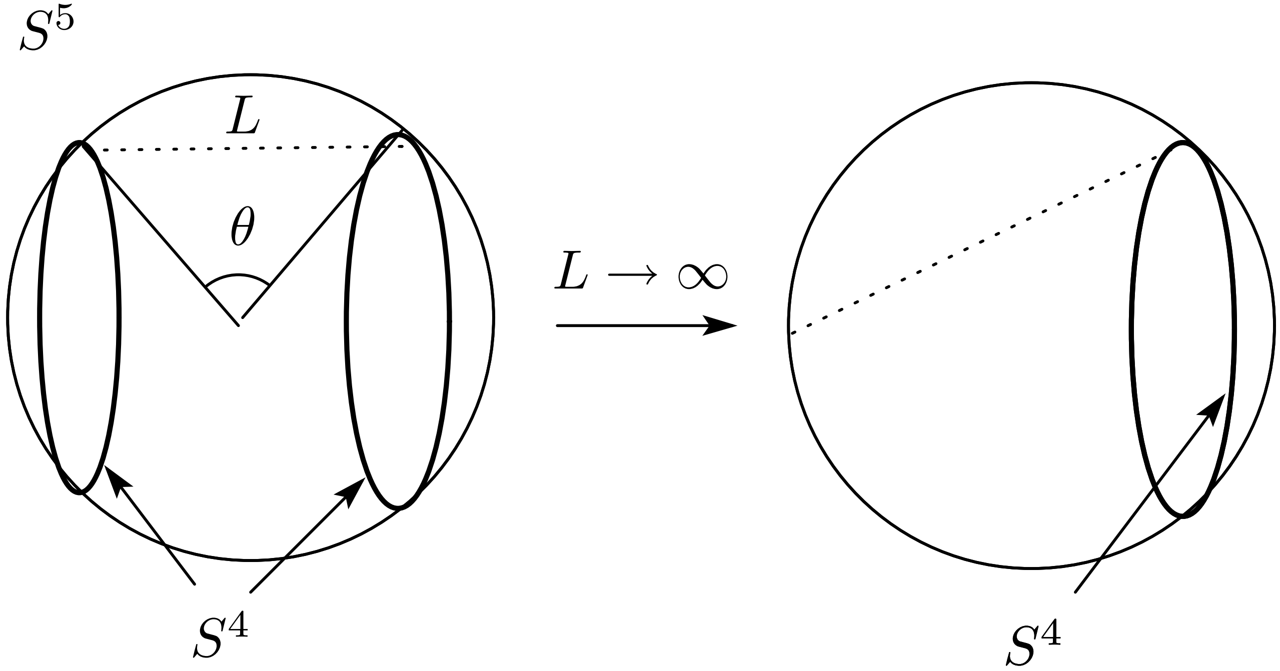
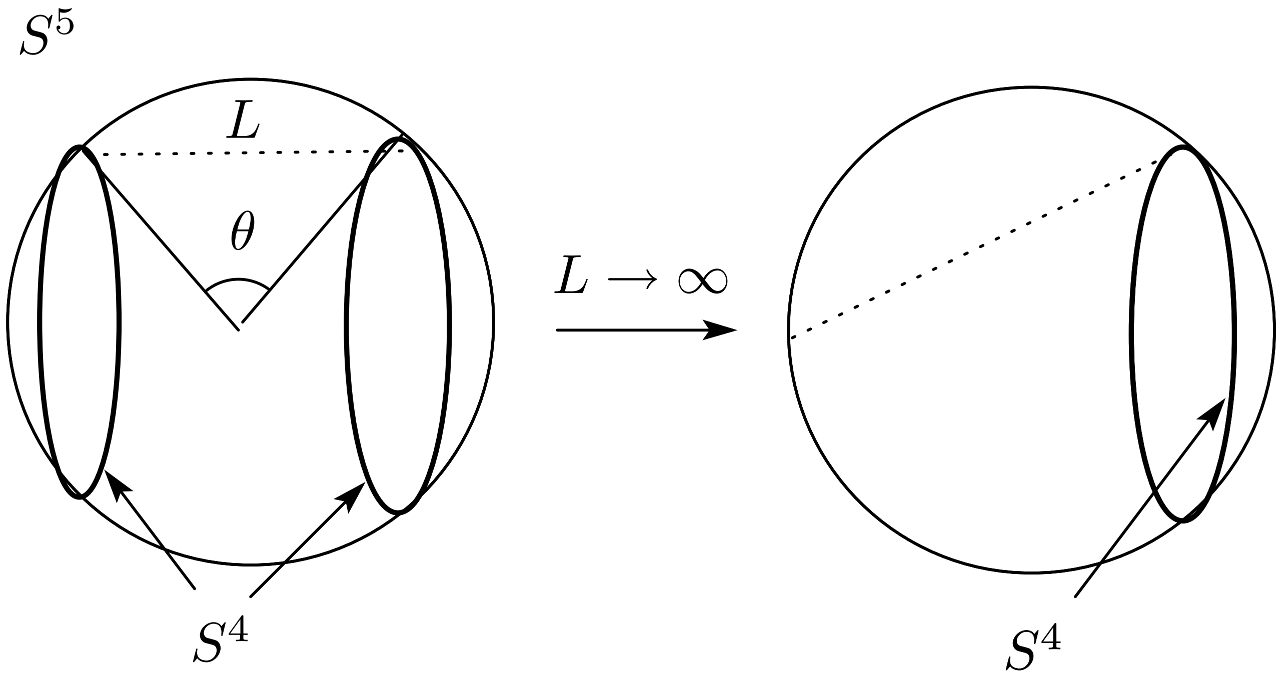
- Bulk: 5-dim AdS or dS space with 2/1 4-dim dS brane (our universe)
- Consistent with observational data even for large extra dimension
- Previous work:
  - flat space brane
  - bulk conformal scalar field
  - conclusion: no CE

We use zeta regularization at full power



Two dS_4 branes in dS_5 background; becomes a one-brane as $L \to \infty$



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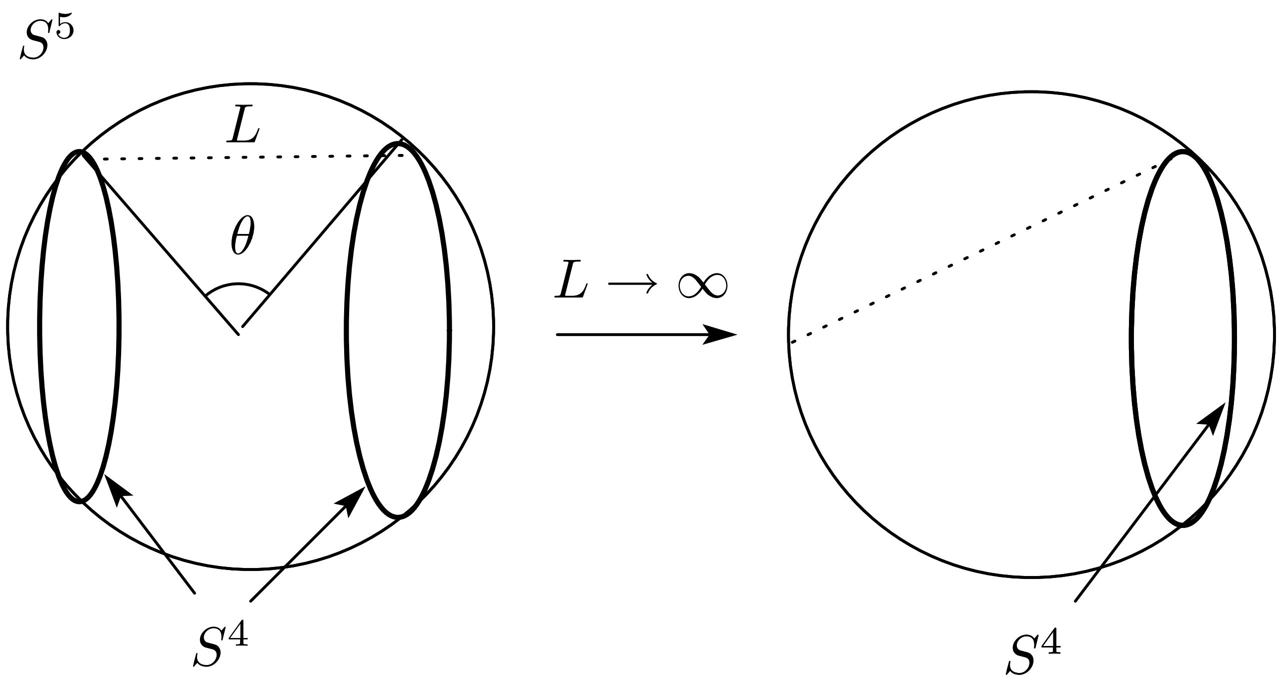

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$\Longrightarrow$ Action for conformally inv massless scalar with scalar-gravit coupling

$$\mathcal{S} = \frac{1}{2} \int d^5x \sqrt{g} \left[-g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi_5 R^{(5)} \phi^2 \right]$$

$\xi_5 = -3/16$ $\quad$ $R^{(5)}$ 5-dim scalar curvature

$\Longrightarrow$ Euclidean metric of the 5-dim AdS bulk

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = \frac{\alpha^2}{\sinh^2 z} \left(dz^2 + d\Omega_4^2 \right) \\ d\Omega_4^2 &= d\xi^2 + \sin^2 \xi \, d\Omega_3^2 \end{aligned}$$

α is the AdS radius, related to cc of AdS bulk

$d\Omega_3$ metric on the 3-sphere

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$$\mathcal{E}_{\text{Cas}} = \frac{\hbar c}{2L\mathcal{R}^4}\zeta\left(-\frac{1}{2}|L_5\right) = -\frac{\hbar c\pi^3}{36L^6}\left[\frac{\pi^2}{315} - \frac{1}{240}\left(\frac{L}{\mathcal{R}}\right)^2 + \mathcal{O}\left(\frac{L}{\mathcal{R}}\right)^4\right]$$

The one-loop effective potential

$$V = \frac{1}{2L\text{Vol}(M_4)} \log\det(L_5/\mu^2) L_5 = -\partial_z^2 - \Delta^{(4)} - \xi_5 R^{(4)} = L_1 + L_4$$

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(b) In the small distance expansion

$$\zeta'(0|L_5) = \frac{\zeta'(-4)\pi^4\mathcal{R}^4}{6L^4} + \frac{\zeta'(-2)\pi^2\mathcal{R}^2}{12L^2} + \frac{1}{24}\left[\zeta_H'(-4,3/2) - \frac{1}{2}\zeta_H'(-2,3/2)\right]\ln\frac{\pi^2\mathcal{R}^2}{L^2}$$

$$+\frac{\zeta'(0)}{6}\left[\zeta_H'(-4,3/2) - \frac{1}{2}\zeta_H'(-2,3/2)\right] + \frac{1}{24}\zeta_H'(-4,3/2)$$

$$+\frac{1}{36}\left[\frac{1}{8}\zeta_H'(-4,3/2) - \frac{1}{3}\zeta_H'(-6,3/2)\right]\frac{L^2}{\mathcal{R}^2} + \mathcal{O}\left(\frac{L^4}{\pi^4\mathcal{R}^4}\right)$$

$$\simeq 0.129652\,\frac{\mathcal{R}^4}{L^4} - 0.025039\,\frac{\mathcal{R}^2}{L^2} - 0.002951\,\ln\frac{\mathcal{R}^2}{L^2} - 0.017956 - 0.000315\frac{L^2}{\mathcal{R}^2} + \cdots$$

The massive case

Lagrangian for a massive scalar field with scalar-gravitational coupling in an AdS background

$$\mathcal{L} = \phi \left(\partial_z^2 + \Delta^{(4)} - m^2 l^2 \sinh^{-2} z + \xi_5 R^{(4)} \right) \phi$$

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(a) Small mass limit (with L not large)

eigenvalues

$$\lambda_n^2 \simeq \frac{\pi^2 n^2}{\mu^2 L^2} + m^2 l^2 \frac{\tanh(\mu L/2)}{\mu L/2}.$$

Mass correction: de Sitter brane in AdS bulk

$$\Delta\zeta'(0|L_5) \simeq \frac{a\rho + a^2\rho^2}{48} - \frac{\pi^2}{144}\left\{\frac{a\rho^2}{2} + [2\zeta'(-4,3/2) - \zeta'(-2,3/2)]\,\rho\right\}$$

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Elizalde, Nojiri, Odintsov, Ogushi, PRD 67 (2003) 063515

C. Supergraviton Theories

- ⟹ Cognola, Elizalde, Zerbini, PLB624 (2005) 70
- ⟹ Cognola, Elizalde, Nojiri, Odintsov, Zerbini, MPLA19 (2004) 1435
- ⟹ Boulanger, Damour, Gualtieri, Henneaux, NPB597 (2001) 127
- ⟹ Sugamoto, Grav. Cosmol. 9 (2003) 91
- ⟹ Arkani-Hamed, Cohen, Georgi, PRL86 (2001) 4757
- ⟹ Arkani-Hamed, Georgi, Schwartz, Ann. Phys. (NY) 305 (2003) 96
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⟹ Allow for non-nearest-neighbor couplings

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operate on the indices n as

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(Δ becomes usual differentiation operator in properly defined continuum limit)

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= \quad -\frac{m^4}{36\pi^2}\log\frac{2^{16}}{3^9} + V_T
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V_T is the sum of all the topological contributions

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$$-\frac{\tilde{M}_0^4}{4\pi^2}\left(\ln\frac{\tilde{M}_0^2}{\mu_R^2} - \frac{3}{2}\right) + \frac{4\tilde{M}_0^4}{3\pi^2}\int_1^\infty du\, G(\tilde{M}_0 ru)\,(u^2-1)^{3/2}$$
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due to the presence of the topological term

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$$V_T \sim \frac{1}{8\pi^2 r^4} \implies V_{eff} > 0 \quad \text{for} \quad mr < \left(\frac{2}{9} \log \frac{2^{16}}{3^9}\right)^{-1/4} \sim 1.4$$

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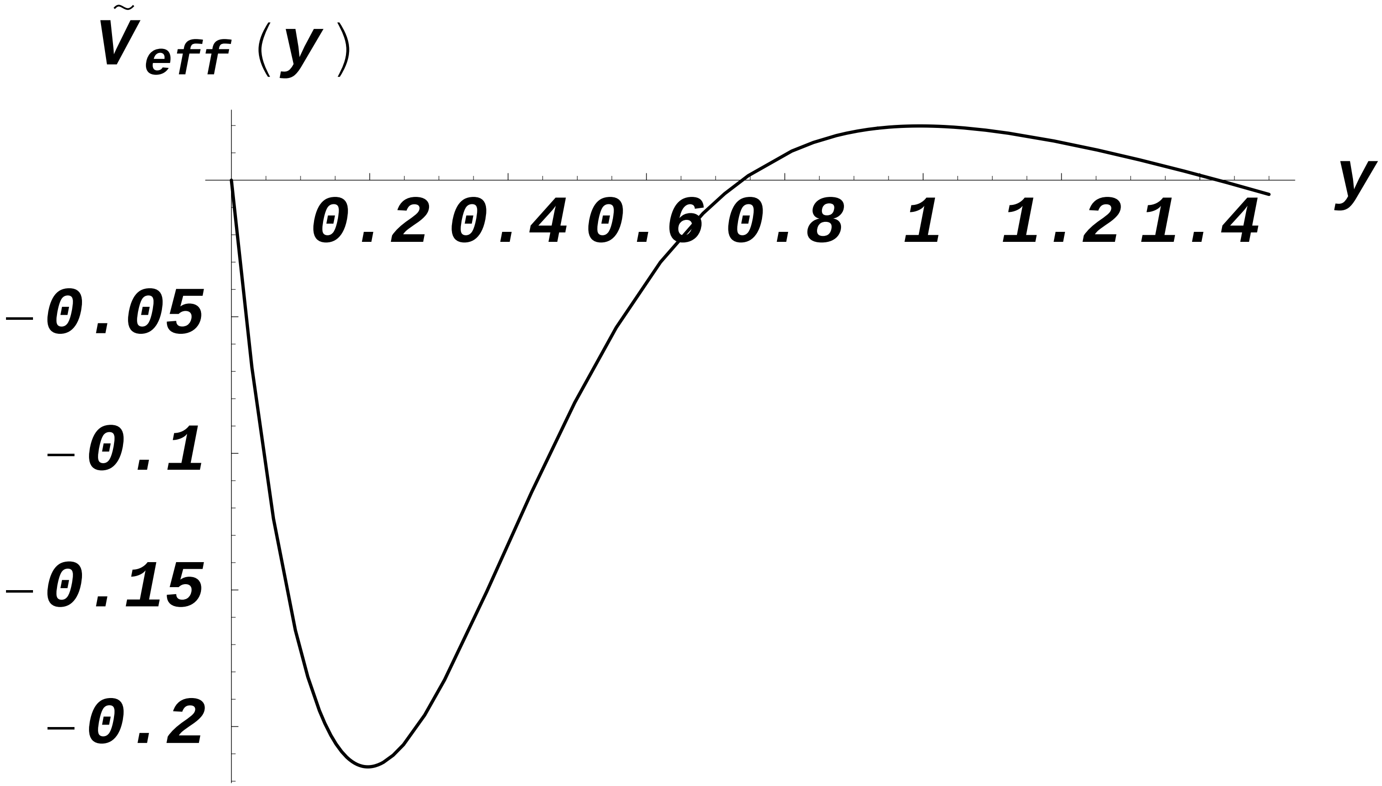

Figure 4: Plot of $\tilde{V}_{eff}(y) \equiv r^4 V_{eff}(r)$ as a function of $y \equiv mr$

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- The value of cc is regulated by the corresponding size of the torus (one can most naturally use the minimum number, N = 3, of copies of bosons and fermions), and can match observational values

Conclusions & Outlook

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Muito Obrigado !