A COSMO-TOPOLOGICAL CASIMIR EFFECT

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http://www.ieec.fcr.es/english/recerca/ftc/eli/eli.htm

1. Fluctuations of the vacuum: the Casimir effect.
2. On the reality of the vacuum fluctuations.
3. The Casimir effect in cosmology.
4. Matching the magnitude and sign of observed values with reasonable models.

EUROGDR05, Barcelona, Cosmology Session, 04/11/2005

Zero point energy

QFT $\langle 0 | H | 0 \rangle$ ← vacuum

Spectrum (normal ordering)

$$H = \left(n + \tfrac{1}{2}\right) \lambda_n \, a_n a_n^{\dagger}$$

$$\langle 0 | H | 0 \rangle = \frac{1}{2} \sum_n \lambda_n = \frac{1}{2} \operatorname{tr} H$$

physical meaning?

$$\infty$$

Regularization + Renormalization

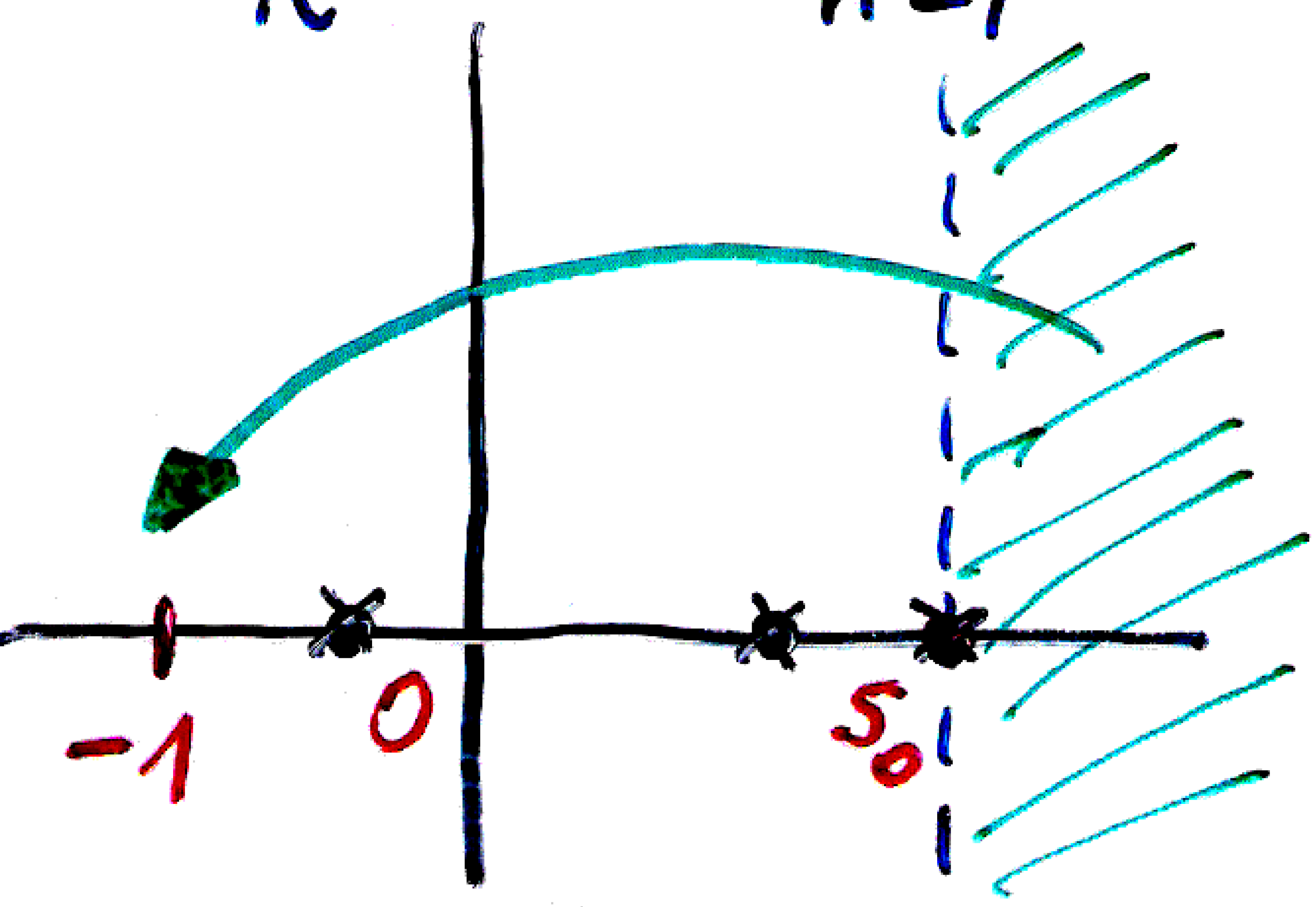
- → cut off
- → dimensional reg.

Direct, math. elegant approaches:

- divergent series
- Borel summability
- ζ-function reg.

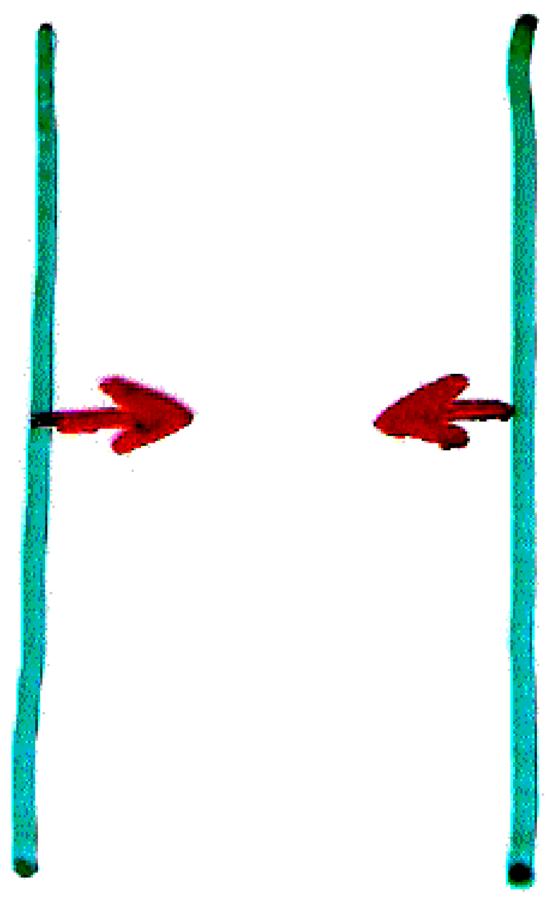
$$\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \zeta_H(s;c) = \sum_{n=0}^{\infty} (n+c)^{-s}$$

$\mathrm{Re}\, s > 1$



$$\langle 0 | H | 0 \rangle = \frac{1}{2} \zeta_H(-1)$$

Casimir effect



vacuum

BC : e.g. periodic
- all kind of fields
- curvature

universal process:
- sonoluminiscence (Schwinger)
- condensed matter (wetting ^3He)
- optical cavities
- direct exp. confirmation (3-exp.)

related:
- van der Waals
- Lipschitz theory (polarizab.)

recent: extract energy from vacuum

dynamical Casimir effect
- two moving mirrors
- particle + moving mirror
- wire ...

→ contribution to the "CC"

dep. a^{-4} (a plate sep.)

ON THE 'REALITY' OF THE ZERO POINT FLUCTS

- The Casimir effect gives no more nor less support for the "reality" of the vacuum fluctuations than other one-loop effects in QED (like vacuum polarization contribution to Lamb shift)

  [R. Jaffe, PRD72 (2005) 021301; hep-th/0503158]

- The Casimir force can be calculated without reference to vacuum fluctuations

Are zero point fluctuations of the vacuum real?

- Schwinger attempted to formulate QED without reference to ZPF
- No one has been able to show that source theory or another S-matrix based approach can provide a complete description of QED to all orders
- In QCD confinement seems to present an insuperable challenge, since quarks and gluons do not appear in the physical S-matrix
- Milonni has recently reformulated all of QED from the point of view of ZPF

THE STANDARD APPROACH

→ Casimir force calculated by computing change in zero point energy of the em field

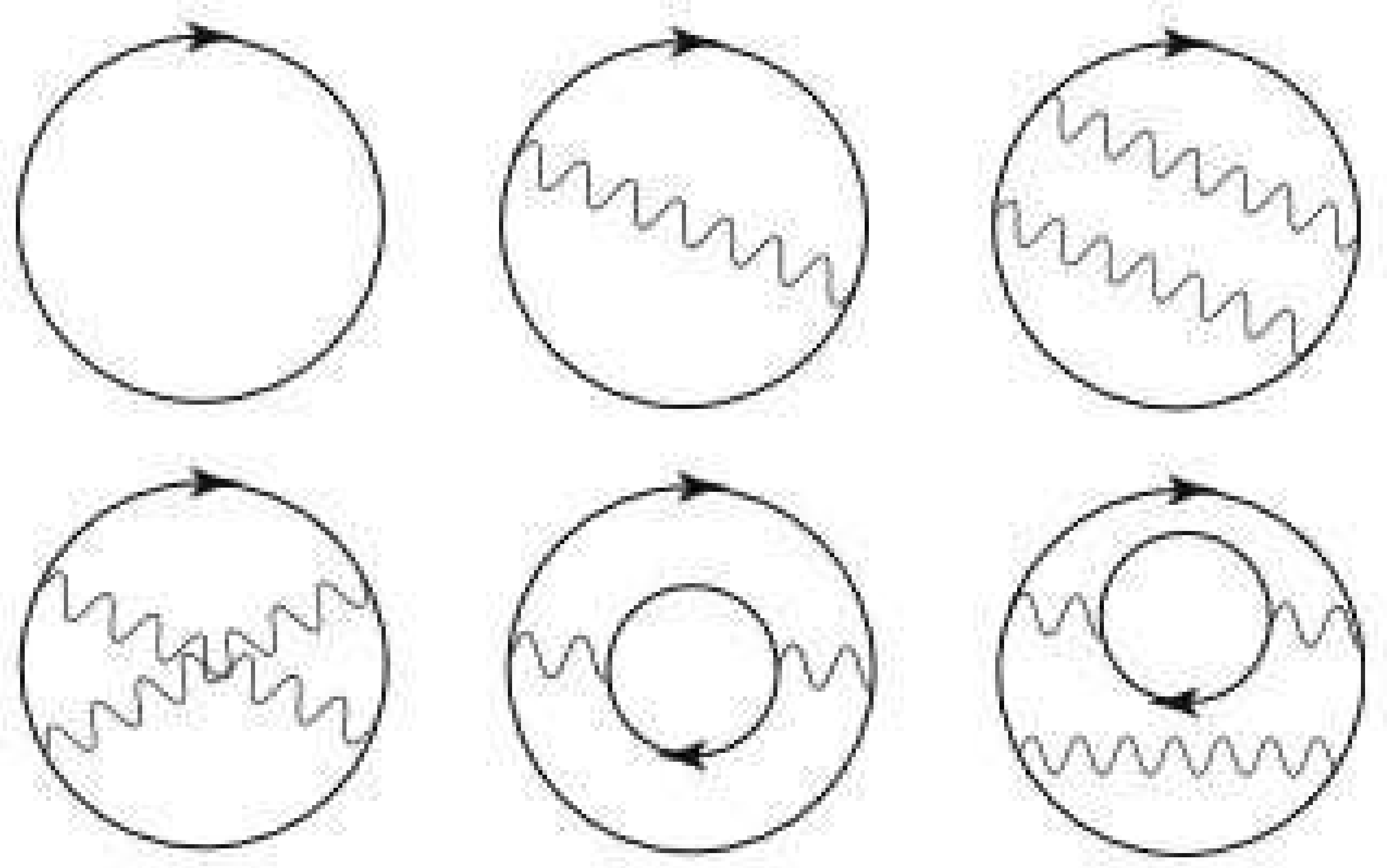


Figure 1: QED graphs contributing to the zero point energy

→ But Casimir effects can be calculated as S-matrix elements: Feynman diagrams with external lines

- In modern language the Casimir energy can be expressed in terms of the trace of the Greens function for the fluctuating field in the background of interest (conducting plates)

$$\mathcal{E} = \frac{\hbar}{2\pi} \mathrm{Im} \int d\omega \omega \, \mathrm{Tr} \int d^3x \, [\mathcal{G}(x, x, \omega + i\epsilon) - \mathcal{G}_0(x, x, \omega + i\epsilon)]$$

$\mathcal{G}$ full Greens function for the fluctuating field
$\mathcal{G}_0$ free Greens function
Trace is over spin

- Moreover
$$\frac{1}{\pi}\mathrm{Im}\int[\mathcal{G}(x,x,\omega+i\epsilon)-\mathcal{G}_0(x,x,\omega+i\epsilon)]=\frac{d\Delta N}{d\omega}$$
change in the density of states due to the background

- So this is a restatement of the Casimir sum over shifts in zero-point energies
$$\frac{1}{2}\sum(\hbar\omega-\hbar\omega_0)$$

$\rightarrow$ Lippman-Schwinger eq. allows full Greens function, $\mathcal{G}$ be expanded as a series in free Green's function, $\mathcal{G}_0$, and the coupling to the external field

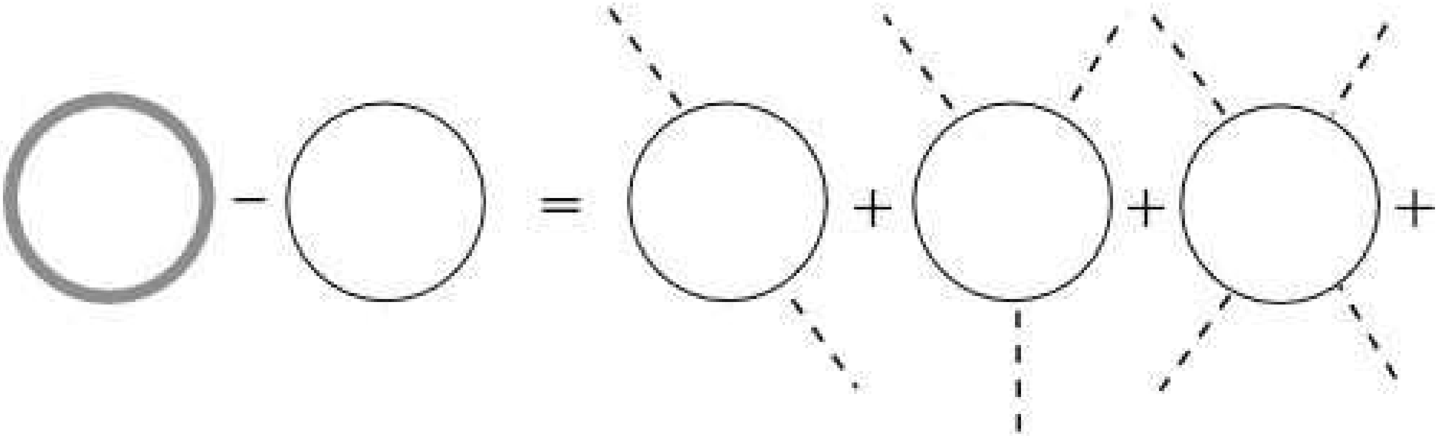


Figure 2: Diagrammatic expansion of the Casimir force: thick (thin) line denotes full (free) Greens function

→ Experimental confirmation of the Casimir effect does not establish the reality of zero point fluctuations

→ Example of similar situation in electrostatics

- The energy of a smooth charge distribution, $\rho(x)$, can be calculated directly

$$\frac{1}{2}\int dx\, dy\, \frac{\rho(x)\rho(y)}{|x-y|}$$

- Or alternatively, from the energy stored in the electric field

$$\frac{1}{8\pi}\int dx\, |\vec{E}(x)|^2$$

- The existence of the latter formula is not an evidence for the “reality” of the electric field
- But propagating electromagnetic waves were discovered
- Look for direct evidence of vacuum fluctuations!

THE VACUUM ENERGY DENSITY & THE CC

The main issue:

Energy ALWAYS gravitates, therefore
the energy density of the vacuum, more precisely,
the vacuum expectation value of the stress-energy tensor

$$\langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu}$$

appears on the rhs of Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G (\tilde{T}_{\mu\nu} - \mathcal{E} g_{\mu\nu})$$

It affects cosmology:
$\tilde{T}_{\mu\nu}$ contribution of excitations above the vacuum
equivalent to a cosmological constant $\lambda = 8\pi G \mathcal{E}$

Recent observations:
M. Tegmark *et al.* [SDSS Collaboration] PRD 2004

$$\begin{aligned} \lambda &= (2.14 \pm 0.13 \times 10^{-3}\ \mathrm{eV})^4 \\ &\sim 4.32 \times 10^{-9}\ \mathrm{erg/cm}^3 \end{aligned}$$

Idea: zero point fluctuations do contribute to the
cosmological constant

COSMO-TOPOLOGICAL CASIMIR EFFECT

- Zeta techniques have been used recently in calculations of the contribution of the vacuum energy of the quantum fields pervading the universe to the cosmological constant (cc)
- Direct calculations of the absolute contributions of the known fields (all couple to gravity) lead to a value which is off by roughly 120 orders of magnitude
$\longrightarrow$ kind of a modern (and very thick!) aether
- Observational tests see nothing (or very little) of it:
$\longrightarrow$ *cosmological constant problem*.
- Very difficult to solve and we do not address this question directly [S. Coleman,...]
- What we do consider —with relative success in some different approaches— is the *additional* contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:
$\longrightarrow$ kind of cosmological Casimir effects
- Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until very recently), we will be left with this incremental value coming from the topology or BCs

- We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in a number of reasonable models involving:

  (a) small and large compactified scales

  (b) dS & AdS worldbranes

  (c) supergravitons

A. A VERY SIMPLE MODEL: large & small dimensions

- Space-time:

$$\mathbf{R^{d+1}} \times \mathbf{T^{p}} \times \mathbf{T^{q}}$$
$$\mathbf{R^{d+1}} \times \mathbf{T^{p}} \times \mathbf{S^{q}}$$
$$\vdots$$

- Scalar field, ϕ, pervading the universe
- ρ_ϕ contribution to ρ_V from this field

$$\rho_\phi = \frac{1}{2}\sum_i \lambda_i = \frac{1}{2}\sum_{\mathbf{k}} \frac{1}{\mu}\left(k^2 + M^2\right)^{1/2}$$

- $\sum_i$ and $\sum_{\mathbf{k}}$ are generalized sums
- μ mass-dim parameter, $\hbar = c = 1$
- M mass of the field arbitrarily small
  (a tiny mass for the field can never be excluded, see L. Parker & A. Raval, PRL86 749 (2001); PRD62 083503 (2000))

For d–open, (p,q)–toroidal universe:

$$\rho_\phi = \frac{\pi^{-d/2}}{2^d\Gamma(d/2)\prod_{j=1}^p a_j \prod_{h=1}^q b_h}\int_0^\infty dk\, k^{d-1} \sum_{\mathbf{n}_p=-\infty}^{\infty}\sum_{\mathbf{m}_q=-\infty}^{\infty}$$

$$\left[\sum_{j=1}^p\left(\frac{2\pi n_j}{a_j}\right)^2 + \sum_{h=1}^q\left(\frac{2\pi m_h}{b_h}\right)^2 + \mathbf{k}_d^2 + M^2\right]^{1/2}$$

$$\simeq \frac{1}{a^p b^q}\sum_{\mathbf{n}_p,\mathbf{m}_q=-\infty}^{\infty}\left(\frac{1}{a^2}\sum_{j=1}^p n_j^2 + \frac{1}{b^2}\sum_{h=1}^q m_h^2 + M^2\right)^{\frac{d+1}{2}+1}$$

For d–open, (p–toroidal, q–spherical) universe:

$$\rho_\phi = \frac{\pi^{-d/2}}{2^d\Gamma(d/2)\ \prod_{j=1}^p a_j\ b^q}\int_0^\infty dk\, k^{d-1} \sum_{\mathbf{n}_p=-\infty}^{\infty}\sum_{l=1}^{\infty} P_{q-1}(l)$$

$$\left[\sum_{j=1}^p\left(\frac{2\pi n_j}{a_j}\right)^2 + \frac{Q_2(l)}{b^2} + \mathbf{k}_d^2 + M^2\right]^{1/2} \simeq$$

$$\frac{1}{a^p b^q}\sum_{\mathbf{n}_p=-\infty}^{\infty}\sum_{l=1}^{\infty} P_{q-1}(l)\left(\frac{4\pi^2}{a^2}\sum_{j=1}^p n_j^2 + \frac{l(l+q)}{b^2} + \frac{M^2}{4\pi^2}\right)^{\frac{d+1}{2}+1}$$

- $P_{q-1}(l)$ is a polynomial in l of degree $q-1$

Regularize: zeta function

$$\zeta(s) = \frac{1}{2}\sum_i \lambda_i^{-s} = \frac{1}{2}\sum_{\mathbf{k}} \left(\frac{k^2+M^2}{\mu^2}\right)^{-s/2}$$

$$\rho_\phi = \zeta(-1)$$

- No further subtraction or renormalization needed here [E.E., J. Math. Phys. **35**, 3308 , 6100 (1994)]

MATHEMATICAL TOOLS

Spectrum of the Hamiltonian op. known explicitly

- Generalized Chowla-Selberg expression [E.E., Commun. Math. Phys. **198**, 83 (1998) E.E., J. Phys. **A30**, 2735 (1997)]

Consider the zeta function ($\Re s > p/2$):

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbf{Z}^p}{}' \left[\frac{1}{2}(\vec{n}+\vec{c})^T A(\vec{n}+\vec{c})+q\right]^{-s} \equiv \sum_{\vec{n}\in\mathbf{Z}^p}{}' \left[Q(\vec{n}+\vec{c})+q\right]^{-s}$$

- Prime means that point $\vec{n} = \vec{0}$ is excluded from sum (not important).
- Gives (analytic continuation of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane
- Exhibits singularities (simple poles) of meromorphic continuation —with corresponding residua— explicitly

Only condition on matrix A: corresponds to (non negative) quadratic form, Q. Vector $\vec{c}$ arbitrary, while q will (for the moment) be a positive constant.

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)}$$

$$\sum_{\vec{m}\in \mathbf{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c}) \left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbf{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbf{Z}^p$ intervene in the sum. That is, if we take an $\vec{m} \in \mathbf{Z}^p$ we must then exclude $-\vec{m}$ (as simple criterion one can, for instance, select those vectors in $\mathbf{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive).

PRECISE CALCULATION OF

THE VACUUM ENERGY DENSITY

Analytic continuation of the zeta function:

$$\zeta(s) = \frac{2\pi^{s/2+1}}{a^{p-(s+1)/2}b^{q-(s-1)/2}\Gamma(s/2)} \sum_{\mathbf{m}_q=-\infty}^{\infty} \sum_{h=0}^{p} \binom{p}{h} 2^h$$

$$\times \sum_{\mathbf{n}_h=1}^{\infty} \left(\frac{\sum_{j=1}^{h} n_j^2}{\sum_{k=1}^{q} m_k^2 + M^2} \right)^{(s-1)/4}$$

$$\times K_{(s-1)/2} \left[\frac{2\pi a}{b} \left(\sum_{j=1}^{h} n_j^2 \right)^{1/2} \left(\sum_{k=1}^{q} m_k^2 + M^2 \right)^{1/2} \right]$$

Yields the vacuum energy density:

$$\rho_\phi = -\frac{1}{a^p b^{q+1}} \sum_{h=0}^{p} \binom{p}{h} 2^h \sum_{\mathbf{n}_h=1}^{\infty} \sum_{\mathbf{m}_q=-\infty}^{\infty} \sqrt{\frac{\sum_{k=1}^{q} m_k^2 + M^2}{\sum_{j=1}^{h} n_j^2}}$$

$$\times K_1 \left[\frac{2\pi a}{b} \left(\sum_{j=1}^{h} n_j^2 \right)^{1/2} \left(\sum_{k=1}^{q} m_k^2 + M^2 \right)^{1/2} \right]$$

Now, from

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)(z/2)^{-\nu}, \qquad z \to 0$$

when M is very small

$$\rho_\phi = -\frac{1}{a^p b^{q+1}} \left\{ M\, K_1\left(\frac{2\pi a}{b} M\right) + \sum_{h=0}^{p} \binom{p}{h} 2^h \right.$$
$$\times \sum_{\mathbf{n}_h=1}^{\infty} \frac{M}{\sqrt{\sum_{j=1}^{h} n_j^2}}\, K_1\left(\frac{2\pi a}{b} M \sqrt{\sum_{j=1}^{h} n_j^2}\right)$$
$$\left. + \mathcal{O}\left[q\sqrt{1+M^2} K_1\left(\frac{2\pi a}{b}\sqrt{1+M^2}\right)\right]\right\}$$

Inserting the $\hbar$ and c factors

$$\rho_\phi = -\frac{\hbar c}{2\pi a^{p+1} b^q}\left[1 + \sum_{h=0}^{p}\binom{p}{h} 2^h \alpha\right] + \mathcal{O}\left[q K_1\left(\frac{2\pi a}{b}\right)\right]$$

α some finite constant (explicit geometrical sum in the limit $M \to 0$).

- Sign may change with BC (e.g., Dirichlet).

MATCHING THE OBSERVATIONAL RESULTS

FOR THE COSMOLOGICAL CONSTANT

$$b \sim l_{P(lanck)}$$

$$a \sim R_{U(niverse)}$$

$$a/b \sim 10^{60}$$

ρ_ϕ	$p = 0$	$p = 1$	$p = 2$	$p = 3$
$b = l_P$	10^{-13}	10^{-6}	1	10^{5}
$b = 10 l_P$	10^{-14}	$[10^{-8}]$	10^{-3}	10
$b = 10^2 l_P$	10^{-15}	(10^{-10})	10^{-6}	10^{-3}
$b = 10^3 l_P$	10^{-16}	10^{-12}	$[10^{-9}]$	(10^{-7})
$b = 10^4 l_P$	10^{-17}	10^{-14}	10^{-12}	10^{-11}
$b = 10^5 l_P$	10^{-18}	10^{-16}	10^{-15}	10^{-15}

Table 1: The vacuum energy density contribution, in units of erg/cm^3, for p large compactified dimensions a, and $q = p + 1$ small compactified dimensions b, $p = 0, \ldots, 3$, for different values of b, proportional to l_P. In brackets, the values that more exactly match the one for the cosmological constant coming from observations, and in parenthesis the otherwise closest approximations.

RESULTS

* Precise coincidence with the observational value for the cosmological constant with $\rho_\phi \longrightarrow [\quad]$

* Approx coincidence $\longrightarrow (\quad)$

* $b \sim$ 10 to 1000 times the Planck length l_P

* $q = p + 1$: (1,2) and (2,3) compactified dimensions, respectively

* Everything dictated by two basic lengths: Planck value and radius of our Universe

* Both situations correspond to a marginally closed universe

To examine $\rightarrow$ couplings in GR

$\rightarrow$ alternative theories

B. MORE AMBITIOUS MODELS: dS & AdS BW

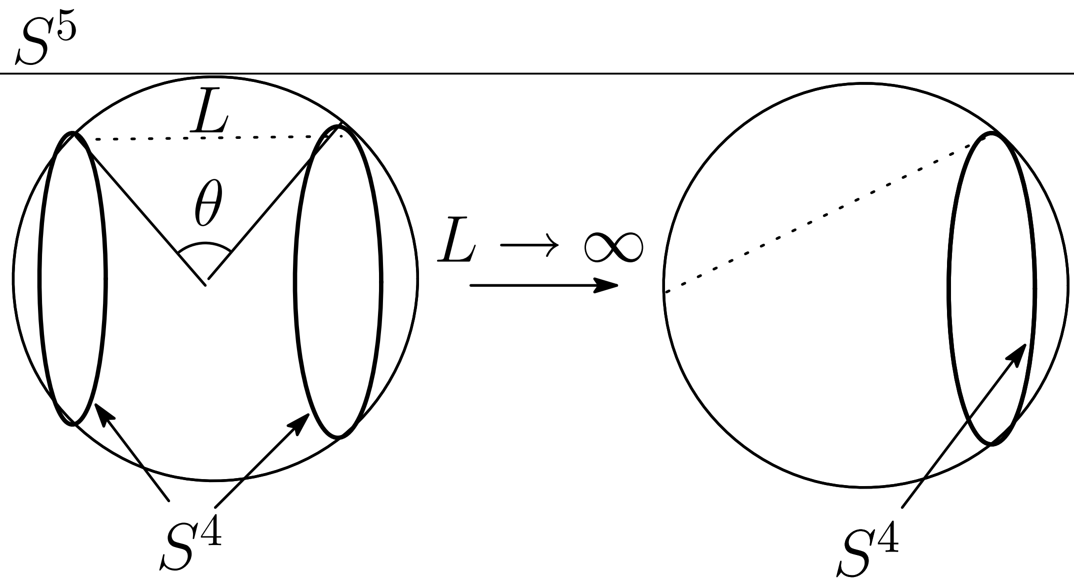
Elizalde, Nojiri, Odintsov, *Late-time cosmology in (phantom) scalar-tensor theory: dark energy and the cosmic speed-up*, PRD70 (2004) 043539

Elizalde, Nojiri, Odintsov, Ogushi, *Casimir effect in dS and AdS braneworlds*, PRD67 (2003) 063515

1. *Braneworld models may help to solve:*
   - the hierarchy problem
   - the cosmological constant problem
2. *Presumably, the bulk Casimir effect will play a role in the construction (radion stabilization) of braneworlds*

- Bulk Casimir effect (effective potential) for a conformal or massive scalar field
- Bulk is a 5-dim AdS or dS space with 2/1 4-dim dS brane (our universe)
- Consistent with observational data even for relatively large extra dimension
- Previous work:
  - flat space brane
  - bulk conformal scalar field
  - conclusion: no CE

We use zeta regularization at full power



Two dS_4 branes in a dS_5 background.
Becomes a one-brane configuration as $L \to \infty$

$\longrightarrow$ Casimir energy density and effective potential for a de Sitter (dS) brane in a five-dimensional anti-de Sitter (AdS) background

$\longrightarrow$ Action for a conformally invariant massless scalar with scalar-gravitational coupling

$$\mathcal{S} = \frac{1}{2} \int d^5x \sqrt{g} \left[-g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi_5 R^{(5)} \phi^2 \right]$$

$\xi_5 = -3/16$
$R^{(5)}$ 5-dim scalar curvature

$\longrightarrow$ Euclidean metric of the 5-dim AdS bulk

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{\alpha^2}{\sinh^2 z} \left(dz^2 + d\Omega_4^2 \right)$$
$$d\Omega_4^2 = d\xi^2 + \sin^2 \xi d\Omega_3^2$$

α is the AdS radius, related to cc of AdS bulk
$d\Omega_3$ metric on the 3-sphere.

CASIMIR ENERGY

(a) One-brane Casimir energy $= 0$

(b) Bulk Casimir energy:

$$\zeta(s|L_5) = \frac{\mu^{-2s}}{6}\sum_{n,l=1}^{\infty}(l+1)(l+2)(2l+3) \times\left[\left(\frac{\pi n}{L}\right)^2 + \mathcal{R}^{-2}\left(l^2+3l+\frac{9}{4}\right)\right]^{-s}$$

L brane separation

$\mathcal{R}$ brane radius

(an imposing zeta function)

Can be written as

$$\zeta(s|L_5) = \frac{\mathcal{R}^{2s}}{6\mu^{2s}}\sum_{n,l=1}^{\infty}2\left(l+\frac{3}{2}\right)\left\{\left[\left(l+\frac{3}{2}\right)^2+\frac{\pi^2n^2\mathcal{R}^2}{L^2}\right]^{1-s} - \left(\frac{\pi^2n^2}{L^2}+\frac{1}{4}\right)\left[\left(l+\frac{3}{2}\right)^2+\frac{\pi^2n^2\mathcal{R}^2}{L^2}\right]^{-s}\right\}$$

$$\equiv \frac{\mathcal{R}^{2s}}{6\mu^{2s}}\left[Z_1(s)+Z_2(s)\right]$$

Hint

$$\partial_x \sum_{l=1}^{\infty}\left((l+x)^2+q\right)^{-s}\Big|_{x=3/2}, \qquad q\equiv\frac{\pi^2n^2\mathcal{R}^2}{L^2},$$

$$\sum_{n=1}^{\infty}\left[(n+c)^2+q\right]^{-s}$$

$$\sim \left(\frac{1}{2}-c\right)q^{-s}+\frac{q^{-s}}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(n+s)}{n!}q^{-n}\zeta_H(-2n,c)$$

$$+\frac{\sqrt{\pi}\,\Gamma(s-1/2)}{2\Gamma(s)}q^{1/2-s}$$

$$+\frac{2\pi^s}{\Gamma(s)}q^{1/4-s/2}\sum_{n=1}^{\infty}n^{s-1/2}\cos(2\pi nc)K_{s-1/2}\left(2\pi n\sqrt{q}\right)$$

We get

$$Z_1(s) = -\frac{1}{2-s}\left(\frac{\pi^2\mathcal{R}^2}{L^2}\right)^{2-s}\zeta(2s-4)-\frac{1}{\Gamma(s-1)}\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(n+s-2)}{n!}$$
$$\times\left(\frac{\pi^2\mathcal{R}^2}{L^2}\right)^{2-n-s}\zeta(2s+2n-4)\zeta_H'(-2n,3/2)$$

$$Z_2(s) = \frac{1}{1-s}\left(\frac{\pi^2\mathcal{R}^2}{L^2}\right)^{1-s}\left[\frac{\pi^2\mathcal{R}^2}{L^2}\zeta(2s-4)+\frac{1}{4}\zeta(2s-2)\right]$$
$$+\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{(-1)^n\Gamma(n+s-1)}{n!}\left(\frac{\pi^2\mathcal{R}^2}{L^2}\right)^{1-n-s}$$
$$\times\left[\frac{\pi^2\mathcal{R}^2}{L^2}\zeta(2s+2n-4)+\frac{1}{4}\zeta(2s+2n-2)\right]\zeta_H'(-2n,3/2)$$

The Casimir energy density (pressure) follows easily

$$\mathcal{E}_{\rm Cas} = \frac{\hbar c}{2LR^4}\zeta\left(-\frac{1}{2}|L_5\right) = -\frac{\hbar c \pi^3}{36 L^6}\left[\frac{\pi^2}{315} - \frac{1}{240}\left(\frac{L}{\mathcal{R}}\right)^2 + \mathcal{O}\left(\frac{L}{\mathcal{R}}\right)^4\right]$$

about ten times larger than the ordinary CE

$$\mathcal{E}_{\rm CE} = -\frac{\hbar c \pi^2}{240\, L^4}$$

which is about 100 dynes / cm^2 at 100 nm

THE ONE LOOP EFFECTIVE POTENTIAL

$$V = \frac{1}{2L{\rm Vol}(M_4)} \log\det(L_5/\mu^2)$$

where

$$L_5 = -\partial_z^2 - \Delta^{(4)} - \xi_5 R^{(4)} = L_1 + L_4$$

$$\log\det L_5 = \sum_{n,\alpha} \log(\lambda_n^2 + \lambda_\alpha^2) = -\zeta'(0|L_5)$$

(a) The one-brane limit $L \longrightarrow \infty$

$$K_t(L_1) \sim \frac{L}{2\sqrt{\pi t}}$$

$$\zeta'(0|L_5) = \frac{1}{3\mathcal{R}}\left[\zeta_H\left(-4,\frac{3}{2}\right) - \frac{1}{4}\zeta_H\left(-2,\frac{3}{2}\right)\right] = 0$$

(b) The small distance expansion

$$
\begin{aligned}
\zeta'(0|L_5) \quad = \quad & \frac{\zeta'(-4)}{6}\,\frac{\pi^4\mathcal{R}^4}{L^4} + \frac{\zeta'(-2)}{12}\,\frac{\pi^2\mathcal{R}^2}{L^2} \\
& + \frac{1}{24}\left[\zeta_H'(-4,3/2) - \frac{1}{2}\zeta_H'(-2,3/2)\right] \ln\frac{\pi^2\mathcal{R}^2}{L^2} \\
& + \frac{\zeta'(0)}{6}\left[\zeta_H'(-4,3/2) - \frac{1}{2}\zeta_H'(-2,3/2)\right] + \frac{1}{24}\zeta_H'(-4,3/2) \\
& + \frac{1}{36}\left[\frac{1}{8}\zeta_H'(-4,3/2) - \frac{1}{3}\zeta_H'(-6,3/2)\right]\frac{L^2}{\mathcal{R}^2} + \mathcal{O}\left(\frac{L^4}{\pi^4\mathcal{R}^4}\right) \\
\simeq\ & 0.129652\,\frac{\mathcal{R}^4}{L^4} - 0.025039\,\frac{\mathcal{R}^2}{L^2} - 0.002951\,\ln\frac{\mathcal{R}^2}{L^2} \\
& -0.017956 - 0.000315\frac{L^2}{\mathcal{R}^2} + \cdots
\end{aligned}
$$

The massive case

Lagrangian for a massive scalar field with scalar-gravitational coupling in an AdS background

$$\mathcal{L} = \phi\left(\partial_z^2 + \Delta^{(4)} - m^2 l^2 \sinh^{-2} z + \xi_5 R^{(4)}\right)\phi$$

in a dS background

$$\mathcal{L} = \phi\left(\partial_z^2 + \Delta^{(4)} - m^2 \cosh^{-2} z + \xi_5 R^{(4)}\right)\phi$$

Effective potential for the massive scalar field

$$
\begin{aligned}
V \quad &= \quad \frac{1}{2L\mathrm{Vol}(M_4)}\log\det(L_5/\mu^2) \\
L_5 \quad &\equiv \quad -\partial_z^2 + m^2 l^2 \sinh^{-2} z - \Delta^{(4)} - \xi_5 R^{(4)} = L_1 + L_4 \quad (AdS) \\
L_5 \quad &\equiv \quad -\partial_z^2 + m^2 \cosh^{-2} z - \Delta^{(4)} - \xi_5 R^{(4)} = L_1 + L_4 \quad (dS)
\end{aligned}
$$

(a) Small mass limit (with L not large)

eigenvalues

$$\lambda_n^2 \simeq \frac{\pi^2 n^2}{\mu^2 L^2} + m^2 l^2 \frac{\tanh(\mu L/2)}{\mu L/2}.$$

mass correction: de Sitter brane in AdS bulk

$$\Delta\zeta'(0|L_5) \simeq \frac{a\rho + a^2\rho^2}{48} - \frac{\pi^2}{144}\left\{\frac{a\rho^2}{2} + \left[2\zeta'(-4,3/2) - \zeta'(-2,3/2)\right]\rho\right\}$$
$$-\frac{\pi^4}{4370}\left[2\zeta'(-4,3/2) - \zeta'(-2,3/2)\right]\rho^2 + \mathcal{O}(m^6),$$

$$a \equiv \frac{\pi^2\mathcal{R}^2}{L^2}, \quad \rho \equiv \frac{m^2 l^2}{\pi^2}\frac{\tanh(L/2l)}{L/2l}$$

(b) Large mass limit (with L not small)

eigenvalues

$$\lambda_n^2 \simeq \frac{\pi^2 n^2 l^2}{L^2} + \frac{2\arctan(\sinh L/2l)}{\sinh(L/2l)} m^2 l^2 + \frac{\pi n m l^2}{L \sinh(L/2l)} + \cdots$$

dominant contribution

$$\zeta(s|L_5) = \frac{L}{2l\sqrt{\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}$$
$$\times\zeta\left(s - \frac{1}{2}\middle| L_4 + 2m^2\frac{\arctan(\sinh(L/2l))}{\sinh(L/2l)}\right) + \cdots$$

$$\zeta'(0|L_5) = -\frac{4m^2 l^3}{3\mathcal{R}}\frac{\arctan(\sinh L/2l)}{\sinh(L/2l)} + \cdots$$ now non-zero!

Casimir effect in de Sitter and Anti-de Sitter braneworlds,
Elizalde, Nojiri, Odintsov, Ogushi, PRD 67 (2003) 063515

C. SUPERGRAVITON THEORIES

Cognola, Elizalde, Zerbini, Multi-(super)graviton theory on topologically non-trivial backgrounds, hep-th/0506082 (PLB to appear)

Cognola, Elizalde, Nojiri, Odintsov, Zerbini, MPLA19 (2004) 1435

Boulanger, Damour, Gualtieri, Henneaux, NPB597 (2001) 127

Sugamoto, Grav. Cosmol. 9 (2003) 91

Arkani-Hamed, Cohen, Georgi, PRL86 (2001) 4757

Arkani-Hamed, Georgi, Schwartz, Ann. Phys. (NY) 305 (2003) 96

Hill, Pokorski, Wang; Damour, Kogan, Papazoglou; Deffayet, Mourad

- We have computed the effective potential for multi-graviton model with supersymmetry
- Bulk is a flat manifold with torus topology $\mathrm{R} \times T^3$ $\longrightarrow$ shown the induced cosmological constant could be positive due to topological contributions
- Previously considered in R^4

Allow for non-nearest-neighbor couplings

Multi-graviton model defined by taking $N-$copies of fields, with graviton $h_{n\mu\nu}$ and Stückelberg fields $A_{n\mu}$, φ_n. Lagrangian of theory generalization of Kan-Shiraishi

$$\mathcal{L} = \sum_{n=0}^{N-1} \Big[-\frac{1}{2}\partial_\lambda h_{n\mu\nu}\partial^\lambda h_n^{\mu\nu} + \partial_\lambda h_{n\mu}^\lambda \partial_\nu h_n^{\mu\nu} - \partial_\mu h_n^{\mu\nu}\partial_\nu h_n + \frac{1}{2}\partial_\lambda h_n \partial^\lambda h_n$$
$$-\frac{1}{2}\left(m^2 \Delta h_{n\mu\nu}\Delta h_n^{\mu\nu} - (\Delta h_n)^2 \right) - 2\left(m\Delta^\dagger A_n^\mu + \partial^\mu \varphi_n \right)\left(\partial^\nu h_{n\mu\nu} - \partial_\mu h_n\right)$$
$$-\frac{1}{2}\left(\partial_\mu A_{n\nu} - \partial_\nu A_{n\mu}\right)\left(\partial^\mu A_n^\nu - \partial^\nu A_n^\mu\right)\Big]$$

Δ, $\Delta^\dagger$ difference operators

operate on the indices n as

$\Delta\phi_n \equiv \sum_{k=0}^{N-1} a_k \phi_{n+k}$, $\Delta^\dagger \phi_n \equiv \sum_{k=0}^{N-1} a_k \phi_{n-k}$, $\sum_{k=0}^{N-1} a_k = 0$,

a_k are N constants and ϕ_n can be identified with periodic fields on a lattice with N sites if periodic boundary conditions, $\phi_{n+N} = \phi_n$, are imposed

(Δ becomes usual differentiation operator in properly defined continuum limit)

When anti-periodic boundary conditions are imposed in the fermionic sector, the situation changes completely with respect to the bosonic one, since the fermionic mass spectrum becomes quite different. The one-loop effective potential in the anti-periodic case is calculated to be

$$V_{eff} = \frac{M_1^4}{4\pi^2}\left(\ln\frac{M_1^2}{\mu_R^2}-\frac{3}{2}\right) - \frac{4M_1^4}{3\pi^2}\int_1^\infty du\, G(M_1 ru)\,(u^2-1)^{3/2}$$
$$-\frac{\tilde{M}_0^4}{4\pi^2}\left(\ln\frac{\tilde{M}_0^2}{\mu_R^2}-\frac{3}{2}\right) + \frac{4\tilde{M}_0^4}{3\pi^2}\int_1^\infty du\, G(\tilde{M}_0 ru)\,(u^2-1)^{3/2}$$
$$-\frac{\tilde{M}_1^4}{8\pi^2}\left(\ln\frac{\tilde{M}_1^2}{\mu_R^2}-\frac{3}{2}\right) + \frac{2\tilde{M}_1^4}{3\pi^2}\int_1^\infty du\, G(\tilde{M}_1 ru)\,(u^2-1)^{3/2}$$
$$= \; -\frac{m^4}{36\pi^2}\log\frac{2^{16}}{3^9} + V_T$$

V_T is the sum of all the topological contributions.

Note that first term on rhs always negative,

but whole effective potential can be positive,

due to the presence of the topological term.

In the regime $mr \ll 1$ one has

$$V_T \sim \frac{1}{8\pi^2 r^4} \implies V_{eff} > 0 \quad \text{for} \quad mr < \left(\frac{2}{9}\log\frac{2^{16}}{3^9}\right)^{-1/4} \sim 1.4$$

while in the opposite regime, $mr \gg 1$, the topological contribution (although still positive) is negligible, and the effective potential remains negative.

Fig. 1: corresponding plot of full effective potential depicted as a function of $y \equiv mr$.

Change of sign in the correct region clearly observed.

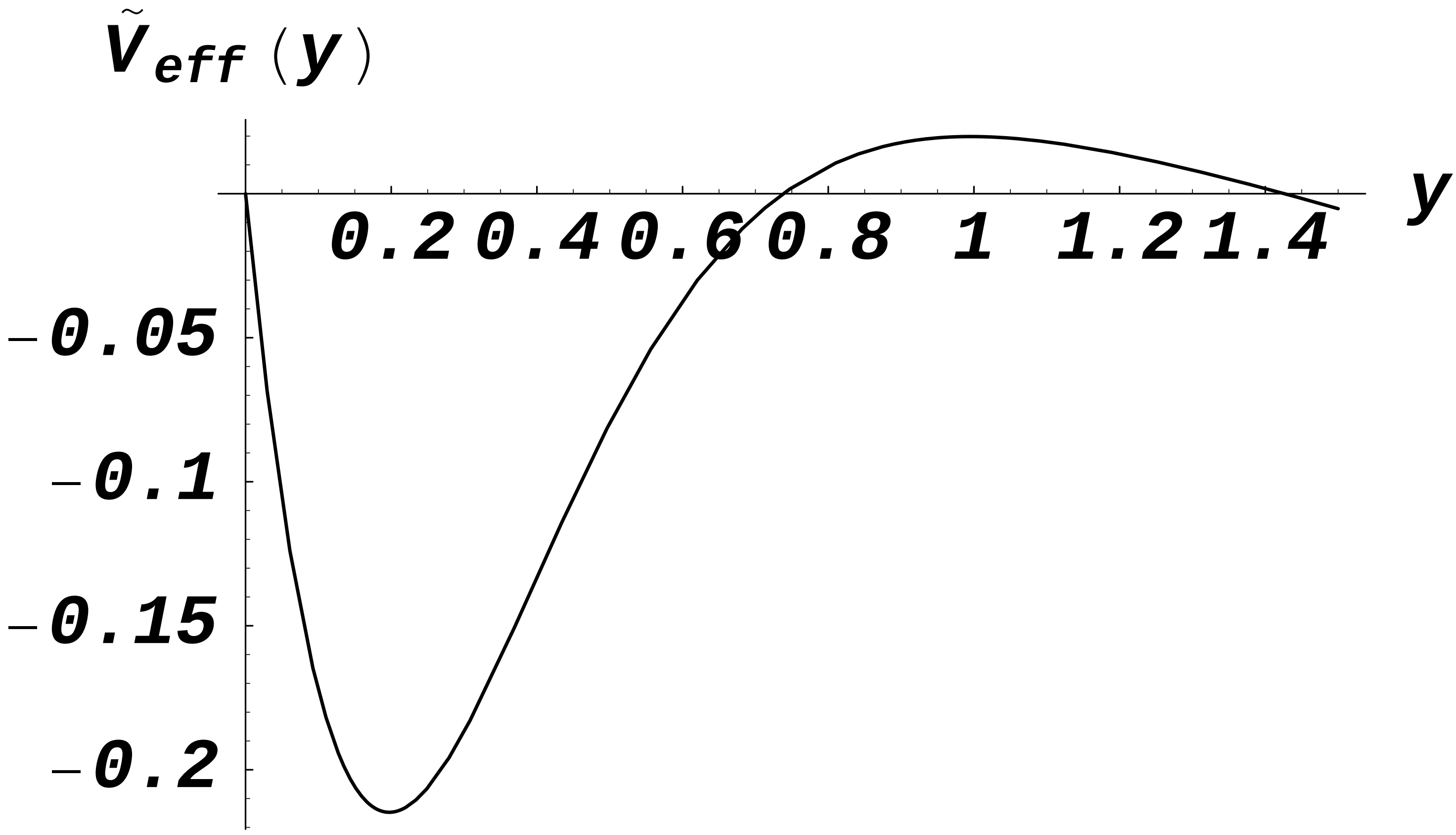


Figure 3: Plot of $\tilde{V}_{eff}(y) \equiv r^4 V_{eff}(r)$ as function of $y \equiv mr$.

- In the multi-graviton model the induced cosmological constant can be positive only if # of massive gravitons is sufficiently large
- In the supersymmetric case the cosmological constant can be positive if one imposes anti-periodic BC in the fermionic sector
- Topological effects discussed may also be relevant in the study of electroweak symmetry breaking in models with a similar type of non-nearest-neighbour couplings, for the deconstruction issue
- Case of the torus topology: top. contributions to the eff. potential have always a fixed sign, depending on the BC one imposes
  - They are negative for periodic fields
  - They are positive for anti-periodic fields.
- Topology provides a mechanism which, in a most natural way, permits to have a positive cc in the multi-supergravity model with anti-periodic fermions
- The value of cc is regulated by the corresponding size of the torus (one can most naturally use the minimum number, $N = 3$, of copies of bosons and fermions), and can match observational values

CONCLUSIONS

- The Casimir Effect is alive and well
- Topology + Vacuum energy may be an important driving force at cosmological scale
- Zeta functions provide a fine and very useful tool for such calculations

THANKS so much for your attention!!