ICE CSIC IEEC

Quantum Vacuum Fluctuations at the Cosmological Level

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“Dynamics and Thermodynamics of Black Holes and Naked Singularities”

Politecnico di Milano - Dipartimento di Matematica, May 12th, 2007

Outline of (an ideal) presentation

- ΨDOs, Zeta Functions, Determinants, and Traces

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- The Chowla-Selberg Expansion Formula (CS) & Extended Expressions (ECS)
- Physics:
  - Quantum vacuum fluctuations
    - $>$ Laboratory (Dynamical CE) J Haro, EE, PRL'06
    - $>$ Actuators (Technology) F Capasso ea, R Onofrio
    - $>$ At Cosmological Scale
  - Non-commutative QFTs (quadratic standard case)
  - New developments (quadratic non-standard cases)

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m $\qquad$ M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that
$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \{f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n\}$$

- $\mathcal{S}'$ space of tempered distributions

- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

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– – ΨDOs are basic tools both in Mathematics & in Physics – –

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[Fredenhagen, Brunetti, . . . R. Wald '06]

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr}\, A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re}\, s > \frac{n}{m} := s_0$$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, Spec $A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \qquad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$

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- By Mellin transform:

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tA}\, t^{s-1}\, dt$$

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  $$s_j = (n-j)/m,\ j \in \mathbb{N},$$

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- If $s_j \in \mathbb{N}$, then $\alpha_j(A)$ is not locally computable

P.B. Gilkey & G. Grubb, CPDE, 1998

G. Cognola, L. Vanzo, S. Zerbini, JMP, 1992

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Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} =$ tr H^{-s}

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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

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- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex s-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C} \backslash \mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik]

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- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = {\sum}'_{\lambda\in\text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n\neq j\geq 0} \alpha_j(A) t^{-s_j} + \sum_{k\geq 1} \beta_k(A) t^k \ln t, \quad t\downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j)\,\text{Res}_{s=s_j}\,\zeta_A(s), \;\; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!}\left[\text{PP}\,\zeta_A(-k) + \psi(k+1)\,\text{Res}_{s=-k}\,\zeta_A(s)\right], \qquad s_j = -k, \;\; k\in\mathbb{N}$$

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$$\text{PP}\,\phi := \lim_{s\to p}\left[\phi(s) - \frac{\text{Res}_{s=p}\,\phi(s)}{s-p}\right]$$

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- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \qquad \mu_0 \geq \mu_1 \geq \cdots$$

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- Definition of the Dixmier trace of T:

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provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

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- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [Connes]

$$\text{Dtr}\, T = \lim_{s\to 1^+} (s-1)\zeta_{T^{-1}}(s)$$

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- Important!: it can be expressed as an integral (local form)

$$\text{res}\, A = \int_{S^*M} \text{tr}\, a_{-n}(x,\xi)\, d\xi$$

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- If $\dim M = n = - \operatorname{ord} A$ (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and $\text{Res}_{s=1} \zeta_A(s) = \frac{1}{n} \text{res } A^{-1}$

The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$
- Only trace one can define in the algebra of ΨDOs (up to multipl const)
- Definition: $\text{res}\, A = 2\, \text{Res}_{s=0}\, \text{tr}\,(A\Delta^{-s})$, Δ Laplacian
- Satisfies the trace condition: $\text{res}\,(AB) = \text{res}\,(BA)$
- Important!: it can be expressed as an integral (local form)

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  with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue)
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- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

Singularities of ζ_A

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- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x,\xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1}=0$:

  $\text{Res}_{s=s_k}\zeta_A(s) = \frac{1}{m}\,\text{res}\,A^{-s_k} = \frac{1}{m}\int_{S^*M} \text{tr}\; a_{-n}^{-s_k}(x,\xi)\,d^{n-1}\xi$

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- Proof. The homog component of degree $-n$ of the corresp power of the principal symbol of A is obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$:

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial\xi^{-1}}\right)^k \left[\xi^{n-k}a^{(k-n)/m}(x,\xi)\right]\Big|_{\xi^{-1}=0}\xi^{-n}$$

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

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- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln\sigma(A,B)]^2\right\}}{2 \text{ ord } A \text{ ord } B \,(\text{ord } A + \text{ord } B)}$$

where $\qquad \sigma(A,B) = A^{\text{ord } B} B^{-\text{ord } A}$

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi]\ \exp\left\{-\int d^Dx\left[\Phi^\dagger(x)(\quad)\Phi(x)+\cdots\right]\right\}$$

Gaussian integration: $\longrightarrow \quad \det\,(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

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- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

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$$\sum_{\lambda=1}^{|D|} \left(\frac{D}{\lambda}\right) \log\Gamma\left(\frac{\lambda}{D}\right) = h\log|D| - \frac{h}{3}\log(2\pi) - \sum_{(a,b,c)} \log a + \frac{2}{3}\sum_{(a,b,c)} \log\left[\theta_1'(0|\alpha)\theta_1'(0|\beta)\right]$$

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- Eta evaluations Dedekind eta function for Im $(\tau) > 0$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q := e^{2\pi i\tau}$$

It is a 24-th root of the discriminant function $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \,|\, a, b \in \mathbb{Z}\}$

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}$$

Properties & Recent Results

$\Longrightarrow$ The C-S formula gives the value of a product of eta functions

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Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau\in\mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\, \theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi nz), \qquad z,t\in\mathbb{C},\ \mathrm{Re}\, t>0$$

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- Truncated sums $\longrightarrow$ asymptotic series

Extended CS Formulas (ECS)

- Consider the zeta function (Re $s > p/2, A > 0,$ Re $q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}$$

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- Case $q \neq 0$ (Re $q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)}$$

$$\times \sum_{\vec{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c}) \left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

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[ECS1]

- Pole: $s = p/2$ Residue:

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive]

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- Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s}\, \Gamma\left(s - \frac{j}{2}\right) \zeta_R(2s-j) + \right.$$
$$\left. 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^\infty \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right) \right]$$

[ECS3d]

QFT in s-t with non-comm toroidal part

- $D-$dim non-commut manifold: $M = \mathbb{R}^{1,d} \bigotimes \mathbb{T}^p_\theta, \quad D = d + p + 1$
  $\mathbb{T}^p_\theta$ a $p-$dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$
  σ_{jk} a real, nonsingular, antisymmetric matrix of ± 1 entries
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- Interest recently, in connection with $M-$theory & string theory
  [Connes,Douglas,Seiberg,Cheung,Chu,Chomerus,Ardalan, . . .]

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- Interest recently, in connection with $M-$theory & string theory [Connes,Douglas,Seiberg,Cheung,Chu,Chomerus,Ardalan, . . .]

- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s) = \frac{V\,\Gamma(s-(d+1)/2)}{(4\pi)^{(d+1)/2}\,\Gamma(s)} \sum_{\vec{n}\in\mathbb{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s}\left[1+\Lambda\theta^{2-2\alpha}Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s}$$

$\alpha = 2$ bos, $\alpha = 3$ ferm, $V = \text{Vol}\,(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j = a_j^{-1/2}$ compactific radii

- After some calculations,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} \left(-\Lambda\theta^{2-2\alpha}\right)^l \zeta_{Q,\vec{0},0}(s+\alpha l - \frac{d+1}{2})$$

for all radii equal to R, with $I(\vec{n}) = \sum_{j=1}^{p} n_j^2$,

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where we use the notation $\zeta_E(s) := \zeta_{I,\vec{0},0}(s)$

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- Classify the different possible cases according to the values of d and $D = d + p + 1$. We obtain, at $s = 0$:

$$\text{For } d = 2k \qquad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = 0 \\ \text{if } D = \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \end{cases}$$

$$\text{For } d = 2k-1 \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} \begin{Bmatrix} \text{finite}, & \text{for } l \leq k \\ 0, & \text{for } l > k \end{Bmatrix} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \\ \\ \text{if } D = 2\alpha l \begin{Bmatrix} \text{pole}, & \text{for } l \leq k \\ \text{finite}, & \text{for } l > k \end{Bmatrix} & \Longrightarrow \quad \zeta_\alpha(0) = \text{pole} \end{cases}$$

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$\Longrightarrow$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$, & specific pole structure

$$\zeta_\alpha(s) = \frac{2^{s-d}\,V}{(2\pi)^{(d+1)/2}\Gamma(s)} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)} (-2^{\alpha}\Lambda\theta^{2-2\alpha})^l \sum_{j=0}^{p-1} (\det A_j)^{-\frac{1}{2}}$$

$$\times \Big[\pi^{j/2} a_{p-j}^{-s-\alpha l+(d+j+1)/2} \Gamma(s+\alpha l-(d+j+1)/2) \zeta_R(2s+2\alpha l-d-j-1)$$

$$+ 4\pi^{s+\alpha l-(d+1)/2} a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{(d+j+1)/2-s-\alpha l}$$

$$\times \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right)\Big]$$

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p \ D	even	odd
odd	(1a) pole / finite ($l \geq l_1$)	(2a) pole / pole
even	(1b) double pole / pole ($l \geq l_1, l_2$)	(2b) pole / double pole ($l \geq l_2$)

– General pole structure of $\zeta_\alpha(s)$, for the possible values of D and p being odd or even. Magenta, type of behavior corresponding to lower values of l; behavior in blue corresponds to larger values of l

Quantum Vacuum Fluct's & the CC

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- Idea: zero point fluctuations can contribute to the cosmological constant Ya.B. Zeldovich '68

- Relativistic field: collection of harmonic oscill's (scalar field)

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- What we do consider —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:

$\Longrightarrow$ kind of cosmological Casimir effect

Cosmo-Topological Casimir Effect

- Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

  * L. Parker & A. Raval, VCDM, vacuum energy density

  * C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two 10^{-2}mm dims, bulk vs brane Susy breaking scales

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  - (c) supergraviton theories (discret dims, deconstr)

A. Simple model: large & small dim's

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  (a tiny mass for the field cannot be excluded, and fits well)

* L. Parker & A. Raval, PRL86 749 (2001); PRD62, 083503 (2000)
* V.G. Gurzadyan & S.-S. Xue, Mod Phys Lett A18, 561 (2003)
* A. Chodos & E. Myers, '85-'86

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- We produce an exact analysis of the one-loop effective action in the 4D, alternative model [EE-Minamitsuji-Naylor, PRD 2007]

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$$V_{4,\text{eff}} = \frac{A_4 - B_4 \ln(\mu^2 \rho_+)}{\rho_+}$$

ρ_+ characterizes the volume modulus

- A_4 and B_4 are functions of the model parameters and the shape modulus (brane tensions)
- Stability determined by sign of B_4 : heat-kernel coeff
- Phenomenological effects on the brane (eff mass of modulus, degree of hierarchy between fundamental energy scales on brane): need to know the value of A_4
- A_4 is not just related to the heat kernel: need to evaluate $\zeta'(0)$
- Accurate evaluation of A_4 is crucial for making physical predictions: hierarchy & CC problems

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where we redefine the terms:

$$\hat{\alpha}(n)=\alpha+\frac{bn}{2a}=\frac{1}{2}+\frac{(1+r)|n|}{2\kappa},\qquad \hat{q}(n)=-\frac{n^2}{\kappa^2}(1-r)^2-1$$

Extended binomial expansion

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[a(m+\alpha)^2+b(m+\alpha)n+cn^2+q\right]^{-s+1}$$

$$=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[a(m+\hat{\alpha})^2+\hat{q}\right]^{-s+1}$$

$$=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(j+s-1)}{\Gamma(s-1)\,j!}\left[a(m+\hat{\alpha})^2\right]^{1-s-j}\,\hat{q}^j$$

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$$=\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}a^{1-s-j}\frac{\Gamma(j+s-1)}{\Gamma(s-1)\,j!}\,\hat{q}^j\,\zeta_H(2s+2j-2,\hat{\alpha}),\qquad\left|\frac{\hat{q}}{a(m+\alpha)^2}\right|<1$$

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$$\zeta(s)\Big|_{n\neq 0}=\frac{g^{2(1-s)}}{\pi}\int d^2x\sum_{j=0}^{\infty}G(j,s)\sum_{n=1}^{\infty}\left[\frac{n^2}{\kappa^2}(1-r)^2+1\right]^j\zeta_H\left(2s+2j-2,\frac{1}{2}+\frac{1+r}{2\kappa}n\right)$$

$$G(j,s)\equiv\frac{2^{-2(j+s-1)}\Gamma(s+j-1)}{\Gamma(s)\,j!}$$

Analytic continuation of the ζ function

$$P(s) = \frac{g^{2(1-s)}}{\pi} \int d^2x \sum_{j=0}^{\infty} G(j,s) \sum_{n=1}^{\infty} \left[\left(\frac{n^2}{\kappa^2}(1-r)^2 + 1 \right)^j \right.$$
$$\left. \times \zeta_H \left(2s+2j-2, \frac{1}{2}+\frac{1+r}{2\kappa} n \right) - F(n,j;s) \right]$$

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$$\zeta(s)\Big|_{n\neq 0} = P(s) + \sum_{j=0}^{\infty} G(j,s)\Delta(j,s),$$ analogously $\zeta_0(s)$

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$$\times \zeta_H \left(2s{+}2j{-}2, \frac{1}{2} {+} \frac{1+r}{2\kappa} n \right) - F(n,j;s) \Bigg]$$

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$$\Delta(j,s) = \frac{2^{-6+2j+2s}\, g^{2(1-s)}}{45(2j+2s-3)\,\pi} \int d^2x\, \kappa^{1+2s} \frac{(1-r)^{2j}}{(1+r)^{1+2j+2s}}$$

$$\times \left[\frac{w_0}{\kappa^4} \zeta_R(2s{-}3) + \frac{w_2(j,s)}{\kappa^2} \zeta_R(2s{-}1) + w_4(j,s) \zeta_R(2s{+}1) \right]$$

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$\Longrightarrow$ Extend our analysis to the cases of

(1) 6 dimensions

(2) a bulk scalar field with self-interactions and other fields in the multiplets appearing in the supergravity model