ICE CSIC IEEC

Non-linear Traces and their Associated Determinants: Theory and Applications

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Outline of this presentation

- ΨDOs, Zeta Function

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- The Casimir Effect and the Cosmological Constant

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m $\qquad$ M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \{f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n\}$$

- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

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– – ΨDOs are basic tools both in Mathematics & in Physics – –

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[Fredenhagen, Brunetti, . . . R. Wald '06]

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as:

$$\zeta_A(s) = \mathrm{tr}\, A^{-s} = \sum_j \lambda_j^{-s}, \quad \mathrm{Re}\, s > \frac{n}{m} := s_0$$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg}\, \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec}\, A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \qquad j = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

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Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr}\; H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1}\, \text{tr}\; e^{-tH}$, Re $s > s_0$

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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

Properties

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- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \, \text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[\text{PP } \zeta_A(-k) + \psi(k+1) \, \text{Res}_{s=-k} \zeta_A(s) \right], \quad s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$\text{PP } \phi := \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p} \right]$$

The Dixmier Trace

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- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \qquad \mu_0 \geq \mu_1 \geq \cdots$$

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- Definition of the Dixmier trace of T:

$$\text{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

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- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [A. Connes]

$$\text{Dtr}\ T = \lim_{s \to 1^+} (s-1) \zeta_{T^{-1}}(s)$$

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- Important!: it can be expressed as an integral (local form)

$$\text{res}\, A = \int_{S^*M} \text{tr}\, a_{-n}(x,\xi)\, d\xi$$

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- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

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- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x,\xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1}=0$:

$$\text{Res}_{s=s_k}\zeta_A(s) = \frac{1}{m}\,\text{res}\;A^{-s_k} = \frac{1}{m}\int_{S^*M}\text{tr}\;a_{-n}^{-s_k}(x,\xi)\,d^{n-1}\xi$$

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- Proof. The homog component of degree $-n$ of the corresp power of the principal symbol of A is obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$:

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial \xi^{-1}}\right)^k \left[\xi^{n-k} a^{(k-n)/m}(x,\xi)\right]\Big|_{\xi^{-1}=0} \xi^{-n}$$

Multiplicative or Noncomm Anomaly or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \, \det_\zeta B$$

$$\det_\zeta(AB) \overset{?}{=} \det_\zeta A \; \det_\zeta B$$

$$\log \det_\zeta = \mathrm{tr}_\zeta \log \quad , \det_\zeta = e^{\mathrm{tr}_\zeta \log}$$

$$\det_\zeta(AB) \overset{1}{=} e^{\mathrm{tr}_\zeta \log(AB)} \overset{2}{=} e^{\mathrm{tr}_\zeta(\log A + \log B)}$$

$$\overset{3}{=} e^{\mathrm{tr}_\zeta \log A + \mathrm{tr}_\zeta \log B} =$$

$$\overset{4}{=} e^{\mathrm{tr}_\zeta \log A} \; e^{\mathrm{tr}_\zeta \log B} =$$

$$\overset{5}{=} \det_\zeta A \cdot \det_\zeta B$$

$[A, B] = 0$ assumed!

Which step is wrong?

tr_ζ is <u>no</u> trace at all

$$\mathrm{tr}_\zeta \, (A_1 + A_2) \neq \mathrm{tr}_\zeta A_1 + \mathrm{tr}_\zeta A_2$$

recall

$$\mathrm{tr}_\zeta A = \zeta_A(s=-1) = \sum_n \lambda_n^{-s} \Big|_{s=-1}$$

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- The multiplicative (or noncommutative) anomaly (defect) is defined as

$$\delta(A,B) = \ln\left[\frac{\det_\zeta(AB)}{\det_\zeta A \ \det_\zeta B}\right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

Multiplicative or Noncomm Anomaly or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det{}_\zeta(AB) \neq \det{}_\zeta A \,\det{}_\zeta B$$

- The multiplicative (or noncommutative) anomaly (defect) is defined as

$$\delta(A,B) = \ln\left[\frac{\det_\zeta(AB)}{\det_\zeta A \;\det_\zeta B}\right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln\sigma(A,B)]^2\right\}}{2\ \text{ord}\,A\ \text{ord}\,B\ (\text{ord}\,A + \text{ord}\,B)}$$

where $\qquad \sigma(A,B) = A^{\text{ord}\,B}B^{-\text{ord}\,A}$

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi]\ \exp\left\{-\int d^D x \left[\Phi^\dagger(x)(\quad)\Phi(x)+\cdots\right]\right\}$$

Gaussian integration: $\longrightarrow \quad \det\,(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

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- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$
- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

Applications: Zero point energy

QFT vacuum to vacuum transition: $\langle 0|H|0\rangle$

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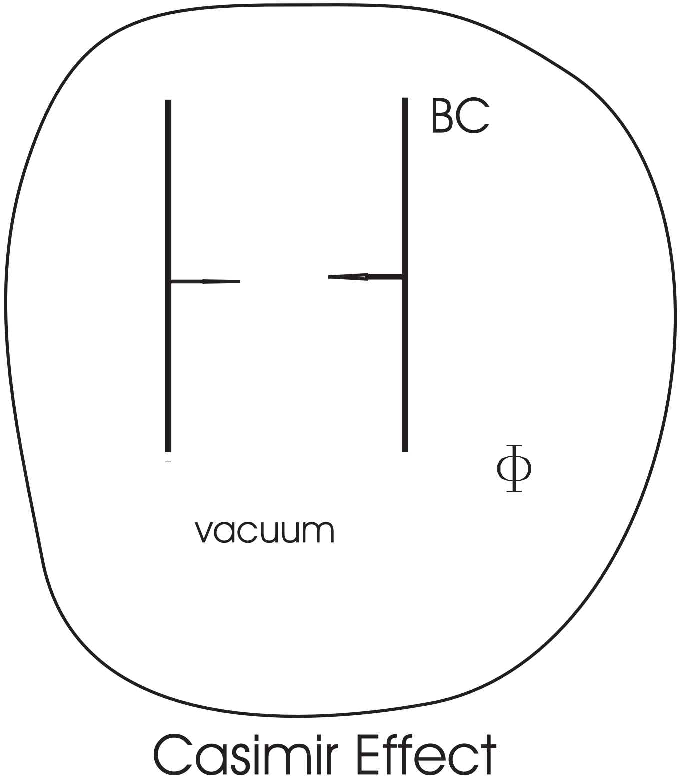
Even then: Has the final value real sense ?

The Casimir Effect

The Casimir Effect

BC e.g. periodic



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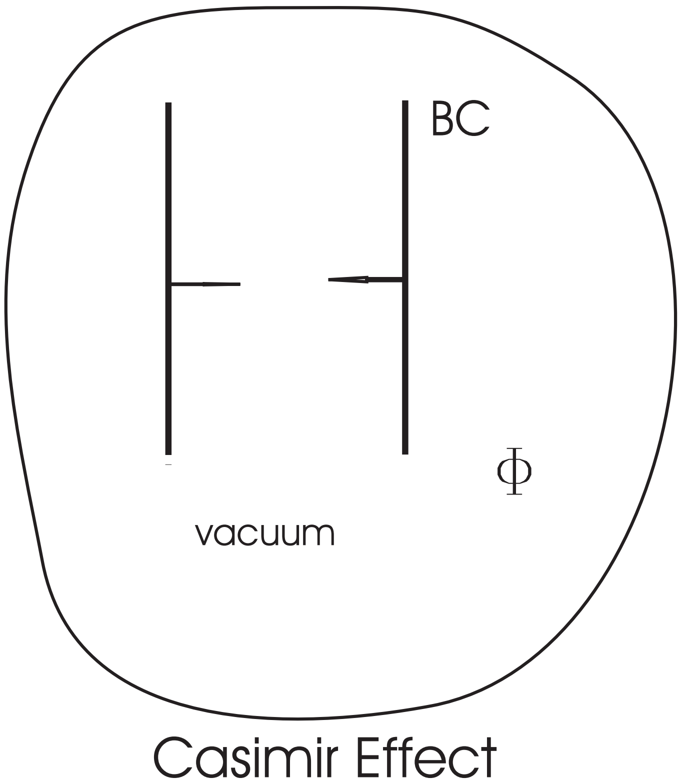

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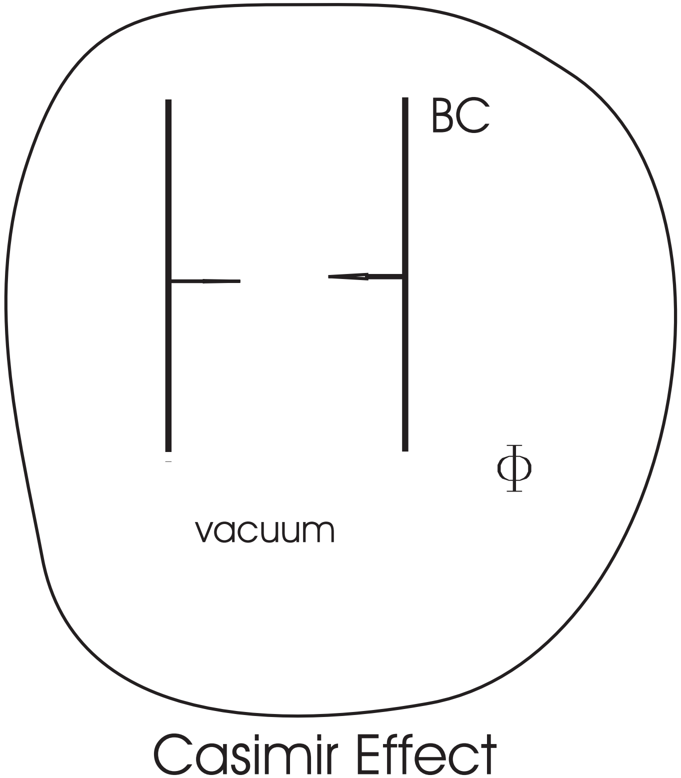

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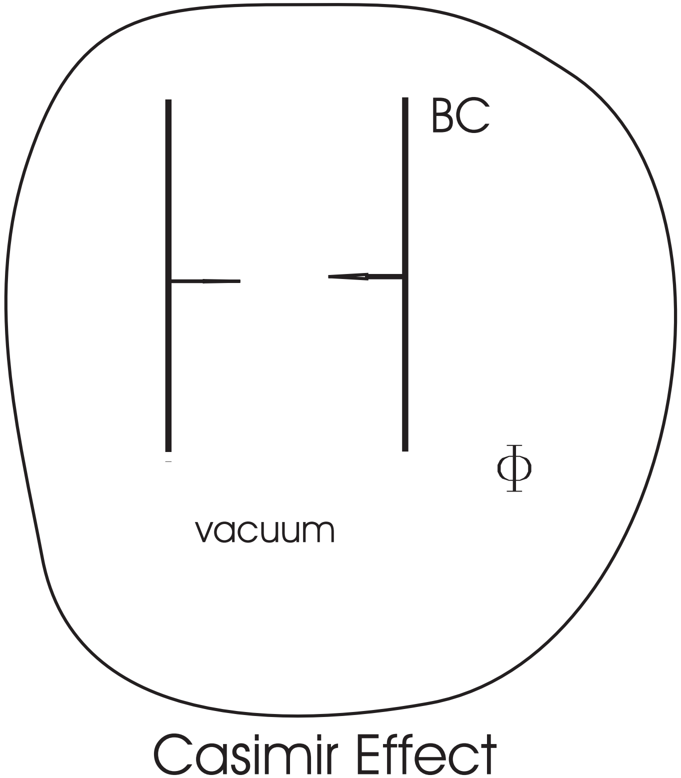

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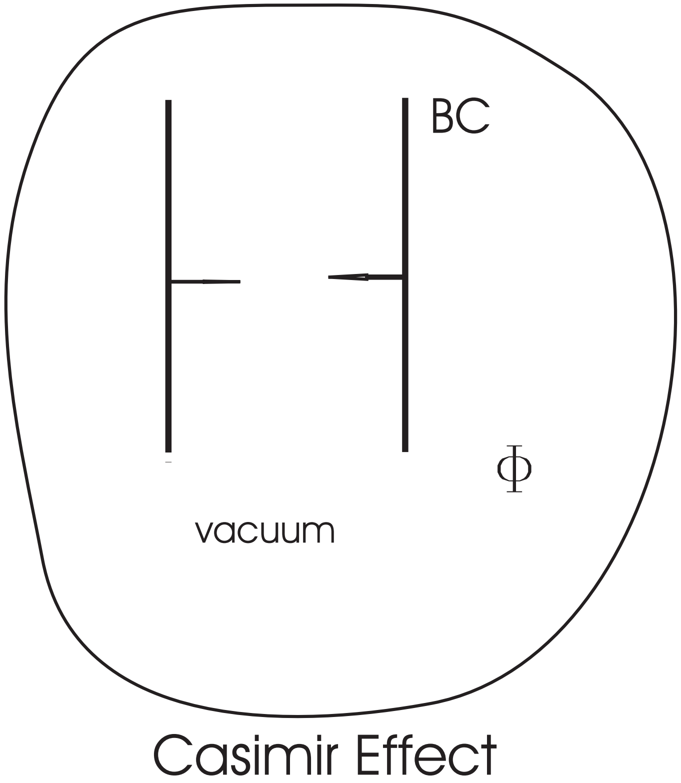

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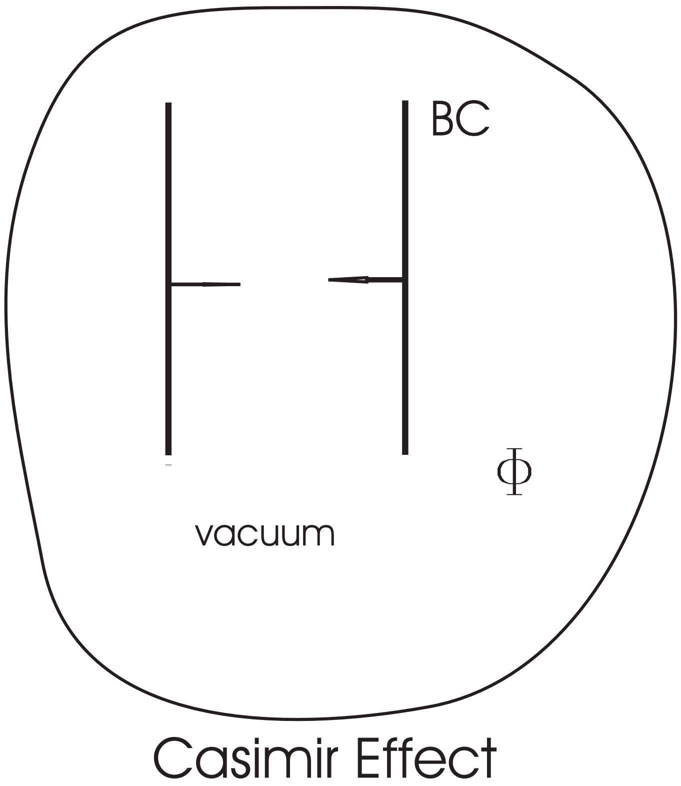

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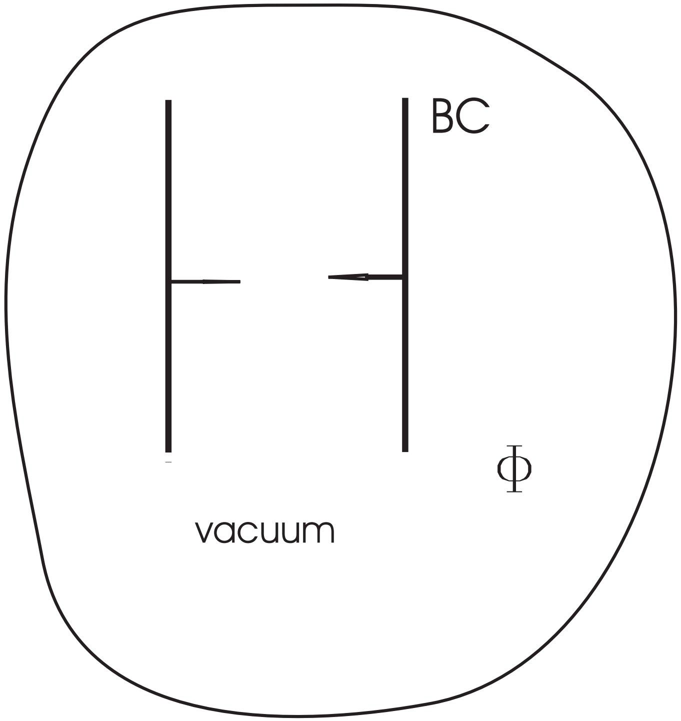

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- Dynamical CE $\Leftarrow$
- Lateral CE
- Extract energy from vacuum
- CE and the cosmological constant $\Leftarrow$

QFT in s-t with non-comm toroidal part

- $D-$dim non-commut manifold: $M = \mathbb{R}^{1,d} \bigotimes \mathbb{T}_\theta^p, \quad D = d + p + 1$

  $\mathbb{T}_\theta^p$ a $p-$dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$

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- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s)=\frac{V\,\Gamma(s-(d+1)/2)}{(4\pi)^{(d+1)/2}\,\Gamma(s)}\sum_{\vec{n}\in\mathbb{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s}\left[1+\Lambda\theta^{2-2\alpha}Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s}$$

$\alpha=2$ bos, $\alpha=3$ ferm, $V=\text{Vol}\,(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n})=\sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j=a_j^{-1/2}$ compactific radii

- After some calculations,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} \left(-\Lambda\theta^{2-2\alpha}\right)^l \zeta_{Q,\vec{0},0}(s+\alpha l-\frac{d+1}{2})$$

for all radii equal to R, with $I(\vec{n}) = \sum_{j=1}^{p} n_j^2$,

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where we use the notation $\zeta_E(s) := \zeta_{I,\vec{0},0}(s)$

e.g., the Epstein zeta function for the standard quadratic form

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- Rich pole structure: pole of Epstein zf at $s = p/2 - \alpha k + (d+1)/2 = D/2 - \alpha k$, combined with poles of Γ, yields a rich pattern of singul for $\zeta_\alpha(s)$

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- Classify the different possible cases according to the values of d and $D = d+p+1$. We obtain, at $s=0$:

$$\text{For } d = 2k \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = 0 \\ \text{if } D = \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \end{cases}$$

$$\text{For } d = 2k-1 \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} \begin{Bmatrix} \text{finite,} & \text{for } l \leq k \\ 0, & \text{for } l > k \end{Bmatrix} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \\ \\ \text{if } D = 2\alpha l \begin{Bmatrix} \text{pole,} & \text{for } l \leq k \\ \text{finite,} & \text{for } l > k \end{Bmatrix} & \Longrightarrow \quad \zeta_\alpha(0) = \text{pole} \end{cases}$$

Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means multiple of 2α)

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$\Longrightarrow$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$,
& specific pole structure

$$\zeta_\alpha(s) = \frac{2^{s-d}\,V}{(2\pi)^{(d+1)/2}\Gamma(s)}\sum_{l=0}^{\infty}\frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)}(-2^{\alpha}\Lambda\theta^{2-2\alpha})^{l}\sum_{j=0}^{p-1}(\det A_j)^{-\frac{1}{2}}$$

$$\times\Big[\pi^{j/2}a_{p-j}^{-s-\alpha l+(d+j+1)/2}\Gamma(s+\alpha l-(d+j+1)/2)\zeta_R(2s+2\alpha l-d-j-1)$$

$$+\,4\pi^{s+\alpha l-(d+1)/2}a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4}\sum_{n=1}^{\infty}\sum_{\vec m_j\in\mathbb{Z}^j}{}' n^{(d+j+1)/2-s-\alpha l}$$

$$\times\left(\vec m_j^t A_j^{-1}\vec m_j\right)^{(s+\alpha l)/2-(d+j+1)/4}K_{(d+j+1)/2-s-\alpha l}\left(2\pi n\sqrt{a_{p-j}\vec m_j^t A_j^{-1}\vec m_j}\right)\Big]$$

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p \ D	even	odd
odd	(1a) pole / finite ($l \geq l_1$)	(2a) pole / pole
even	(1b) double pole / pole ($l \geq l_1, l_2$)	(2b) pole / double pole ($l \geq l_2$)

General pole structure of $\zeta_\alpha(s)$, for the possible values of D and p being odd or even. Magenta, type of behavior corresponding to lower values of l; behavior in blue corresponds to larger values of l

Quantum Vacuum Fluct's & the CC

- The main issue: S.A. Fulling et. al., hep-th/070209v2

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- Idea: zero point fluctuations can contribute to the cosmological constant Ya.B. Zeldovich '68

- Relativistic field: collection of harmonic oscill's (scalar field)

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- What we do consider —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:

  $\Longrightarrow$ kind of cosmological Casimir effect

Cosmo-Topological Casimir Effect

- Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

  * L. Parker & A. Raval, VCDM, vacuum energy density
  * C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two 10^{-2}mm dims, bulk vs brane Susy breaking scales
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- We show (with different examples) that this value acquires the correct order of magnitude —corresponding to the one coming from the observed acceleration in the expansion of our universe— in some reasonable models involving:

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  - (b) dS & AdS worldbranes
  - (c) supergraviton theories (discret dims, deconstr)

Summary:

ΨDO ζ function

[M Atiyah, I Singer, A Connes, M Kontsevich, ...]

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$\swarrow$ $\downarrow$ $\searrow$

Casimir Eff: NEM, Dyn NC QFT CC, Dark E, acc U

[EE et al] [S Hawking]