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Zeta Function Regularization in Casimir Effect Calculations & J.S. Dowker's Contribution

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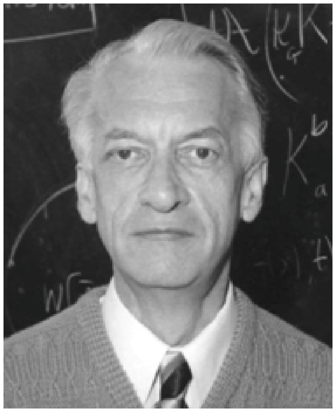
Outline

- The Zero Point Energy
- Operator Zeta Functions in Math Physics: Origins
- J. S. Dowker & S. W. Hawking Contributions
- Pseudodifferential Operator (ΨDO)
- Existence of ζ_A for A a ΨDO
- $\zeta-$Determinant, Properties
- Wodzicki Residue, Multiplicative Anomaly or Defect
- Extended CS Series Formulas (ECS)
- QFT in space-time with a non-commutative toroidal part
- Future Perspectives: Operator regularization & so on
- With THANKS to:

  S Carloni, G Cognola, J Haro, S Nojiri, S Odintsov,
  D Sáez-Gómez, A Saharian, P Silva, S Zerbini, . . .



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Even then: Has the final value real sense ?

Operator Zeta F's in MΦ: Origins

- The Riemann zeta function $\zeta(s)$ is a function of a complex variable, s. To define it, one starts with the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges for all complex values of s with real Re $s > 1$, and then defines $\zeta(s)$ as the analytic continuation, to the whole complex $s-$plane, of the function given, Re $s > 1$, by the sum of the preceding series.

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- Zeta function encoding the eigenvalues of the Laplacian of a compact Riemannian manifold for the case of a compact region of the plane

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- Used this to define the determinant of a positive self-adjoint operator A (the Laplacian of a Riemannian manifold in their application) with eigenvalues $a_1, a_2, ...,$ and in this case the zeta function is formally the trace

$$\zeta_A(s) = \text{Tr}\,(A)^{-s}$$

the method defines the possibly divergent infinite product

$$\prod_{n=1}^{\infty} a_n = \exp[-\zeta_A{}'(0)]$$

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- Abstract
  The effective Lagrangian and vacuum energy-momentum tensor $< T^{\mu\nu} >$ due to a scalar field in a de Sitter space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form $(\Box^2 + \xi R + m^2)\varphi = 0$. If $\xi = 1/6$ and $m = 0$, the renormalized $< T^{\mu\nu} >$ equals $g^{\mu\nu}(960\pi^2 a^4)^{-1}$, where a is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.

Effective Lagrangian and energy-momentum tensor in de Sitter space

J. S. Dowker and Raymond Critchley
Department of Theoretical Physics, The University, Manchester, 13, England
(Received 29 October 1975)

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I. INTRODUCTION

In a previous paper[1] (to be referred to as I) the effective Lagrangian $\mathfrak{L}^{(1)}$ due to single-loop diagrams of a scalar particle in de Sitter space was computed. It was shown to be real and was evaluated as a principal-part integral. The regularization method used was the proper-time one due to Schwinger[2] and others. We now wish to consider the same problem but using different techniques. In particular, we wish to make contact with the work of Candelas and Raine,[3] who first discussed the same problem using dimensional regularization.

Some properties of the various regularizations as applied to the calculation of the vacuum expectation value of the energy-momentum tensor have been contrasted by DeWitt.[4] We wish to pursue some of these questions within the context of a definite situation.

II. GENERAL FORMULAS: REGULARIZATION METHODS

We use exactly the notation of I, which is more or less standard, and begin with the expression for $\mathfrak{L}^{(1)}$ in terms of the quantum-mechanical propagator, $K(x'',x',\tau)$,

$$\mathfrak{L}^{(1)}(x')=-\tfrac{1}{2}i\lim_{x''\to x'}\int_0^\infty d\tau\,\tau^{-1}K(x'',x',\tau)e^{-im^2\tau}+X(x')\,. \tag{1}$$

There are two points regarding this expression which need some further discussion. Firstly, if we adopt the proper-time regularization method so that the infinities appear only when the τ integration, which is the final operation, is performed, then we can take the coincidence limit, $x''=x'$, through into the integrand. Further, since the regularized expression is continuous across the light cone, it does not matter how we let x'' approach x'. Secondly, the term X does not have to be a constant, but it should integrate to give a metric-independent contribution to the total action, $W^{(1)}$.

The Schwinger-DeWitt procedure is to derive an expression for K, either in closed form or as a general expansion to powers of τ, then to effect the coincidence limit in (1), and finally to perform the τ integration. This was the approach adopted in I. We proceed now to give a few more details.

We assume that we are working on a Riemannian space, $\mathfrak{M}$, of dimension d. The coincidence limit $K(x,x,\tau)$ can be expanded,[5]

$$K(x,x,\tau)=i(4\pi i\tau)^{-d/2}\sum_{n=0}^\infty a_n(x)(i\tau)^n\,, \tag{2}$$

where the a_n are scalars constructed from the curvature tensor on $\mathfrak{M}$ and whose functional form is independent of d. The manifold $\mathfrak{M}$ must not have boundaries, otherwise other terms appear in the expansion.

The expansion (2) is substituted into (1) to yield

$$\mathfrak{L}^{(1)}(x)=\tfrac{1}{2}i(4\pi)^{-d/2}\sum_n a_n(x)\int_0^\infty(i\tau)^{n-d/2-1}e^{-im^2\tau}d\tau\,. \tag{3}$$

The infinite terms are those for which $n\leqslant d/2$ (for d even) or $n\leqslant(d-1)/2$ (for d odd). For $d=4$, e.g. space-time, there are three infinite terms. These terms are removed by renormalization; the details are given by DeWitt.[4]

Another popular regularization technique is dimensional regularization.[6] In this method the dimension, d, is considered to be complex and all expressions are defined in a region of the d plane where they converge. The infinities appear when an analytic continuation to $d=4$ is performed to regain the physical quantities. This idea was originally developed for use in flat-space (i.e., Lorentz-invariant) situations for the momentum

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Communications in
Mathematical
Physics

Zeta Function Regularization of Path Integrals in Curved Spacetime

S. W. Hawking

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, England

Abstract. This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to n dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fith dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This energy momentum tensor has an anomalous trace.

1. Introduction

The purpose of this paper is to describe a technique for obtaining finite values to path integrals for fields (including the gravitational field) on a curved spacetime background or, equivalently, for evaluating the determinants of differential operators such as the four-dimensional Laplacian or D'Alembertian. One forms a gemeralised zeta function from the eigenvalues λ_n of the operator

$$\zeta(s)=\sum_n \lambda_n^{-s}\ . \tag{1.1}$$

In four dimensions this converges for $\mathrm{Re}(s)>2$ and can be analytically extended to a meromorphic function with poles only at $s=2$ and $s=1$. It is regular at $s=0$. The derivative at $s=0$ is formally equal to $-\sum_n \log\lambda_n$. Thus one can define the determinant of the operator to be $\exp(-d\zeta/ds)|_{s=0}$.

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m $\qquad$ M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \{f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n\}$$

- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

– An elliptic ΨDO is one with an elliptic symbol

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– – ΨDOs are basic tools both in Mathematics & in Physics – –

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[K Fredenhagen, R Brunetti, . . . R Wald '06, R Haag EPJH35 '10]

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr}\, A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re}\, s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of A, $s_0 = \dim M / \text{ord}\, A$ abscissa of converg of $\zeta_A(s)$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg}\, \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec}\, A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \qquad j = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$$

Definition of Determinant

H ΨDO operator $\{\varphi_i, \lambda_i\}$ spectral decomposition

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Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr } H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \text{tr } e^{-tH}$, Re$s > s_0$

Derivative: $\zeta_H'(0) = -\sum_{i\in I} \ln \lambda_i$

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Determinant: Ray & Singer, '67

$$\det\nolimits_\zeta H = \exp\left[-\zeta_H'(0)\right]$$

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Determinant: Ray & Singer, '67

$$\det_\zeta H = \exp\left[-\zeta'_H(0)\right]$$

Weierstrass def: subtract leading behavior of λ_i in i, as $i \to \infty$, until series $\sum_{i\in I} \ln \lambda_i$ converges $\Longrightarrow$ non-local counterterms !!

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As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \text{tr } e^{-tH}$, Re$s > s_0$

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Determinant: Ray & Singer, '67

$$\det_\zeta H = \exp\left[-\zeta_H'(0)\right]$$

Weierstrass def: subtract leading behavior of λ_i in i, as $i \to \infty$, until series $\sum_{i\in I} \ln \lambda_i$ converges $\Longrightarrow$ non-local counterterms !!

C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

Properties

- The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut
- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex s-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C}\backslash\mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik]
- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum\nolimits'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum\nolimits_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum\nolimits_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j)\,\text{Res}_{s=s_j}\,\zeta_A(s),\ s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!}\left[\text{PP}\,\zeta_A(-k) + \psi(k+1)\,\text{Res}_{s=-k}\,\zeta_A(s)\right], \qquad s_j = -k,\ k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!}\,\text{Res}_{s=-k}\,\zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$\text{PP}\,\phi := \lim_{s\to p}\left[\phi(s) - \frac{\text{Res}_{s=p}\,\phi(s)}{s-p}\right]$$

The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$
- Only trace one can define in the algebra of ΨDOs (up to multipl const)
- Definition: $\text{res}\, A = 2\, \text{Res}_{s=0}\, \text{tr}\,(A\Delta^{-s})$, Δ Laplacian
- Satisfies the trace condition: $\text{res}\,(AB) = \text{res}\,(BA)$
- Important!: it can be expressed as an integral (local form)

  $$\text{res}\, A = \int_{S^*M} \text{tr}\, a_{-n}(x,\xi)\, d\xi$$

  with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue)
- If $\dim M = n = -\,\text{ord}\, A$ (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and $\text{Res}_{s=1}\zeta_A(s) = \frac{1}{n}\,\text{res}\, A^{-1}$
- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

Singularities of ζ_A

- A complete determination of the meromorphic structure of some zeta functions in the complex plane can be also obtained by means of the Dixmier trace and the Wodzicki residue

- Missing for full descript of the singularities: residua of all poles

- As for the regular part of the analytic continuation: specific methods have to be used (see later)

- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x, \xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1} = 0$:

  $$\text{Res}_{s=s_k} \zeta_A(s) = \frac{1}{m} \text{res}\, A^{-s_k} = \frac{1}{m} \int_{S^*M} \text{tr}\, a_{-n}^{-s_k}(x, \xi)\, d^{n-1}\xi$$

- Proof. The homog component of degree $-n$ of the corresp power of the principal symbol of A is obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1} = 0$:

$$a_{-n}^{-s_k}(x, \xi) = \left(\frac{\partial}{\partial \xi^{-1}} \right)^k \left[\xi^{n-k} a^{(k-n)/m}(x, \xi) \right] \Big|_{\xi^{-1}=0} \xi^{-n}$$

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det_\zeta(AB) \neq \det_\zeta A \, \det_\zeta B$$

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- The multiplicative (or noncommutative) anomaly (defect) is defined as

$$\delta(A,B) = \ln \left[\frac{\det_\zeta(AB)}{\det_\zeta A \ \det_\zeta B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

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- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2 \ \text{ord}\, A \ \text{ord}\, B \ (\text{ord}\, A + \text{ord}\, B)}$$

where $\qquad \sigma(A,B) = A^{\text{ord}\, B} B^{-\text{ord}\, A}$

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi]\ \exp\left\{-\int d^D x \left[\Phi^\dagger(x)(\quad)\Phi(x)+\cdots\right]\right\}$$

Gaussian integration: $\longrightarrow \quad \det\left(\quad\right)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

$\det(AB)$ or $\det A \cdot \det B$?

- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$

- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau \in \mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \quad z,t \in \mathbb{C},\ \text{Re}\, t > 0$$

- Higher dimensions: Poisson summ formula (Riemann)

$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p} \widetilde{f}(\vec{m})$$

$\widetilde{f}$ Fourier transform

[Gelbart + Miller, BAMS '03, Iwaniec, Morgan, ICM '06]

- Truncated sums $\longrightarrow$ asymptotic series

Extended CS Formulas (ECS)

- Consider the zeta function (Re$s > p/2, A > 0,$ Re$q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}$$

prime: point $\vec{n}=\vec{0}$ to be excluded from the sum
(inescapable condition when $c_1 = \cdots = c_p = q = 0$)

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- Case $q \neq 0$ ($\text{Re}\, q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)}$$

$$\times \sum_{\vec{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c}) \left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

[ECS1]

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[ECS1]

- Pole: $s = p/2$ Residue:

$$\text{Res}_{s=p/2}\,\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}(\det A)^{-1/2}$$

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- Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s}\, \Gamma\left(s-\frac{j}{2}\right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right)\right]$$

[ECS3d]

QFT in s-t with non-comm toroidal part

- $D-$dim non-commut manifold: $M = \mathbb{R}^{1,d} \bigotimes \mathbb{T}^p_\theta, \quad D = d + p + 1$

  $\mathbb{T}^p_\theta$ a $p-$dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$

  σ_{jk} a real, nonsingular, antisymmetric matrix of ± 1 entries

  θ the non-commutative parameter.

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- Interest recently, in connection with $M-$theory & string theory [Connes,Douglas,Seiberg,Cheung,Chu,Chomerus,Ardalan, . . .]

- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s) = \frac{V\,\Gamma(s-(d+1)/2)}{(4\pi)^{(d+1)/2}\,\Gamma(s)} \sum_{\vec{n}\in\mathbb{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s} \big[1+\Lambda\theta^{2-2\alpha} Q(\vec{n})^{-\alpha}\big]^{(d+1)/2-s}$$

$\alpha = 2$ bos, $\alpha = 3$ ferm, $V =$ Vol $(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j = a_j^{-1/2}$ compactific radii

- After some calculations,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} \,(-\Lambda\theta^{2-2\alpha})^l \,\zeta_{Q,\vec{0},0}(s+\alpha l-\frac{d+1}{2})$$

for all radii equal to R, with $I(\vec{n}) = \sum_{j=1}^{p} n_j^2$,

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where we use the notation $\zeta_E(s) := \zeta_{I,\vec{0},0}(s)$

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- Classify the different possible cases according to the values of d and $D = d + p + 1$. We obtain, at $s = 0$:

For $d = 2k$ $\quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = 0 \\ \text{if } D = \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \end{cases}$

For $d = 2k - 1$ $\quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} \left\{ \begin{array}{ll} \text{finite,} & \text{for } l \leq k \\ 0, & \text{for } l > k \end{array} \right\} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \\ \\ \text{if } D = 2\alpha l \left\{ \begin{array}{ll} \text{pole,} & \text{for } l \leq k \\ \text{finite,} & \text{for } l > k \end{array} \right\} & \Longrightarrow \quad \zeta_\alpha(0) = \text{pole} \end{cases}$

– Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means multiple of 2α)

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– Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\overline{\dot{2\alpha}}$ means multiple of 2α)

$\Longrightarrow$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$,
& specific pole structure

$$\zeta_\alpha(s) = \frac{2^{s-d}\,V}{(2\pi)^{(d+1)/2}\Gamma(s)}\sum_{l=0}^{\infty}\frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)}(-2^{\alpha}\Lambda\theta^{2-2\alpha})^{l}\sum_{j=0}^{p-1}(\mathrm{det}A_j)^{-\frac{1}{2}}$$

$$\times\Big[\pi^{j/2}a_{p-j}^{-s-\alpha l+(d+j+1)/2}\Gamma(s+\alpha l-(d+j+1)/2)\zeta_R(2s+2\alpha l-d-j-1)$$

$$+\,4\pi^{s+\alpha l-(d+1)/2}a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4}\sum_{n=1}^{\infty}\sum_{\vec{m}_j\in\mathbb{Z}^j}{}'\,n^{(d+j+1)/2-s-\alpha l}$$

$$\times\left(\vec{m}_j^{\,t}A_j^{-1}\vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4}K_{(d+j+1)/2-s-\alpha l}\left(2\pi n\sqrt{a_{p-j}\vec{m}_j^{\,t}A_j^{-1}\vec{m}_j}\right)\Big]$$

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$$\times\Big[\pi^{j/2}a_{p-j}^{-s-\alpha l+(d+j+1)/2}\Gamma(s+\alpha l-(d+j+1)/2)\zeta_R(2s+2\alpha l-d-j-1)$$

$$+4\pi^{s+\alpha l-(d+1)/2}a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4}\sum_{n=1}^{\infty}\sum_{\vec{m}_j\in\mathbb{Z}^j}{}' n^{(d+j+1)/2-s-\alpha l}$$

$$\times\left(\vec{m}_j^t A_j^{-1}\vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l}\left(2\pi n\sqrt{a_{p-j}\vec{m}_j^t A_j^{-1}\vec{m}_j}\right)\Big]$$

p \ D	even	odd
odd	(1a) pole / finite ($l \geq l_1$)	(2a) pole / pole
even	(1b) double pole / pole ($l \geq l_1, l_2$)	(2b) pole / double pole ($l \geq l_2$)

– General pole structure of $\zeta_\alpha(s)$, for the possible values of D and p being odd or even. Magenta, type of behavior corresponding to lower values of l; behavior in blue corresponds to larger values of l

Future Perspectives: Oper Regulariz

Operator Regularization (OR) approach

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Operator Regularization (OR) approach

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- OR does not cure the non-predictability problem of non-renormalizability, but advantage that the initial Lagrangian need not be extended with addition of extra terms
- The OR scheme is governed by the identity:
$$H^{-m} = \lim_{\epsilon \to 0} \frac{d^n}{d\epsilon^n} \left[1 + \left(1 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \ldots + \alpha_n \epsilon^n \right) \frac{\epsilon^n}{n!} H^{-\epsilon - m} \right]$$
α_i's are arbitrary, and it is enough that the degree of regularization is equal to the loop order, n

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Further Extension

- OR was first introduced in the context of the Schwinger approach, which is known to be equivalent to the Feynman one

- OR of the logarithm in the Schwinger approach

$$\ln H = - \lim_{\epsilon \to 0} \frac{d^n}{d\epsilon^n} \left(\frac{\epsilon^{n-1}}{n!} H^{-\epsilon} \right)$$

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  A Rebhan, Phys Rev D39, 3101 (1989)

  its naive application to obtain finite amplitudes breaks unitarity

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  L Culumovic, M Leblanc, R B Mann, D G C McKeon and T N Sherry, Phys Rev D41, 514 (1990)

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THANK YOU!