ICE CSIC IEEC

ζ - FUNCTION METHODS
&
QUANTUM VACUUM FLUCTUATIONS

EMILIO ELIZALDE

ICE/CSIC & IEEC, UAB, Barcelona

"Quantum Theory and Symmetries - 5", Valladolid, July 22, 2007

Outline of this presentation

- Euler and the Zeta Function

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- ΨDOs, Zeta Functions, Determinants, and Traces

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- The Casimir Effect and the Cosmological Constant

Euler and the Zeta Function

- How did Euler discover the zeta function? [R Ayoub, Am Math Month 81 (1974) 1067]. The harmonic series

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

has an infinite sum. Euler: what about the 'prime harmonic series'

$$PH = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$

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- Euler considered $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$

- Provided s is bigger than 1, you can split it up

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- Idea: s closer to 1, first sum increases without bound. Key step is the celebrated equation (p prime)

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- For any p prime and any $s>1$, set $x=1/p^s$ to give:

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- Euler multiplied together these inf sums: his inf product as single inf sum:

$$\frac{1}{p_1^{k_1 s}\cdots p_n^{k_n s}}$$

$p_1,\ldots,p_n$ primes $k_1,\ldots,k_n$ positive integers, each such combination occurs exactly once, rhs just rearrangement of $\zeta(s)$.

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- Euler's infinite product formula for $\zeta(s)$ marked the beginning of analytic number theory.

- Dirichlet modified the zeta function: primes separated into categories, depending on the remainder when divided by k:

$$L(s,\chi) = \frac{\chi(1)}{1^s} + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \frac{\chi(4)}{4^s} + \cdots$$

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- Conditions: (i) $\chi(mn)=\chi(m)\chi(n)$ for any m,n
(ii) $\chi(n)=\chi(n+k),\ \forall n$
(iii) $\chi(n)=0$ if n,k have a common factor (iv) $\chi(1)=1$

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- Any function $L(s,\chi)$, where s real number greater than 1 and χ character, is known as a Dirichlet L-series. Euler zeta function is a special case: $\chi(n) = 1$ for all n. Another ex. $\chi(n) = \mu(n)$ (Möbius)

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- Subsequent generalizations: allow s and $\chi(n)$ to be complex.
  The celebrated Riemann zeta function, Hurwitz, Epstein, etc.
  Many results about prime numbers were proven: L-series provide a powerful tool for the study of the primes [Keith Devlin]

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m $\qquad$ M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x,\xi) \right| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n \}$$

- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

– An elliptic ΨDO is one with an elliptic symbol

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–– ΨDOs are basic tools both in Mathematics & in Physics ––

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[Fredenhagen, Brunetti, . . . R. Wald ’06]

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr}\, A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re}\, s > \frac{n}{m} := s_0$$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg}\ \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec}\ A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \qquad j = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

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Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $Re\, s > 1$ (& analytic cont)

Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} =$ tr H^{-s}

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1}$ tr e^{-tH}, Re $s > s_0$

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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

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- Asymptotic expansion for the heat kernel:

$$\text{tr}\ e^{-tA} = \sum'_{\lambda\in \text{Spec}\ A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n\neq j\geq 0} \alpha_j(A) t^{-s_j} + \sum_{k\geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j)\,\text{Res}_{s=s_j}\,\zeta_A(s),\ s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!}\left[\text{PP}\ \zeta_A(-k) + \psi(k+1)\,\text{Res}_{s=-k}\,\zeta_A(s)\right], \qquad s_j = -k,\ k\in\mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!}\,\text{Res}_{s=-k}\,\zeta_A(s), \quad k\in\mathbb{N}\backslash\{0\}$$

$$\text{PP}\ \phi := \lim_{s\to p}\left[\phi(s) - \frac{\text{Res}_{s=p}\,\phi(s)}{s-p}\right]$$

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- Satisfies the trace condition: $\text{res}\,(AB) = \text{res}\,(BA)$
- Important!: it can be expressed as an integral (local form)

$$\text{res}\, A = \int_{S^*M} \text{tr}\; a_{-n}(x,\xi)\; d\xi$$

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- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

Multiplicative or Noncomm Anomaly or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

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$$\delta(A,B) = \frac{\text{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2 \, \text{ord}\, A \, \text{ord}\, B \, (\text{ord}\, A + \text{ord}\, B)}$$

where $\qquad \sigma(A,B) = A^{\text{ord}\, B} B^{-\text{ord}\, A}$

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi]\ \exp\left\{-\int d^Dx \left[\Phi^\dagger(x)(\quad)\Phi(x)+\cdots\right]\right\}$$

Gaussian integration: $\longrightarrow \quad \det\left(\quad\right)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \longrightarrow \begin{pmatrix} A & \\ & B \end{pmatrix}$$

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- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

The Chowla-Selberg Expansion Formula: Basics

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^\infty q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau\in\mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \qquad z,t\in\mathbb{C},\ \text{Re}\, t > 0$$

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- Truncated sums $\longrightarrow$ asymptotic series

Extended CS Formulas (ECS)

- Consider the zeta function (Re $s > p/2, A > 0,$ Re $q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}$$

prime: point $\vec{n}=\vec{0}$ to be excluded from the sum

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[ECS3d]

Zero point energy

QFT vacuum to vacuum transition: $\langle 0 | H | 0 \rangle$

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Even then: Has the final value real sense ?

The Casimir Effect

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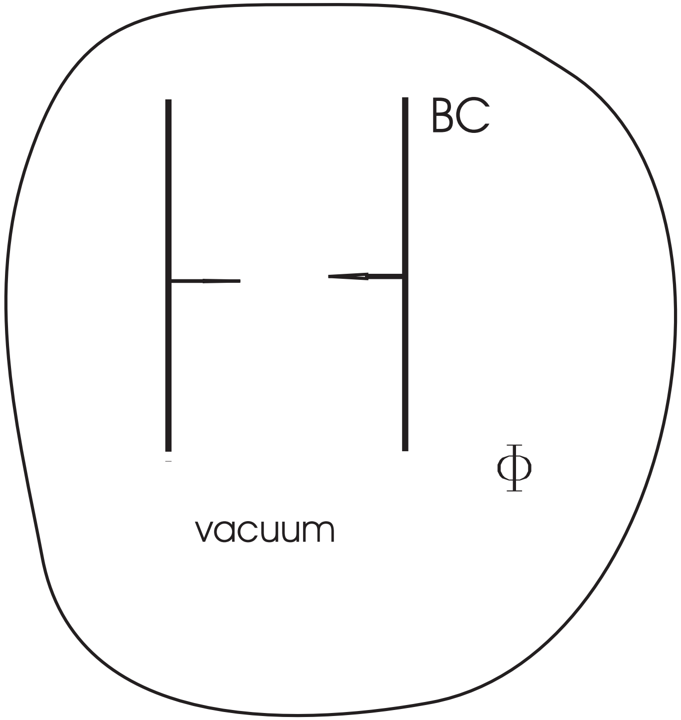

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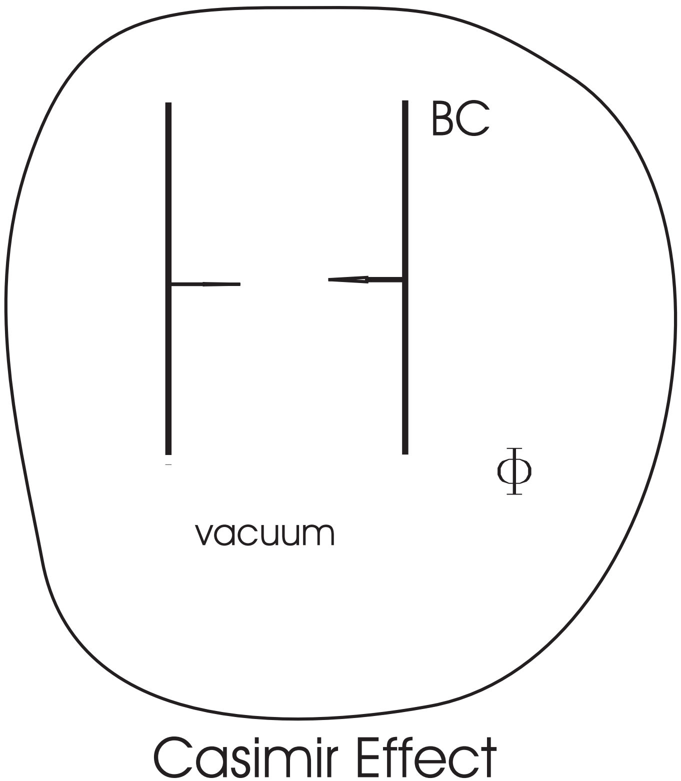

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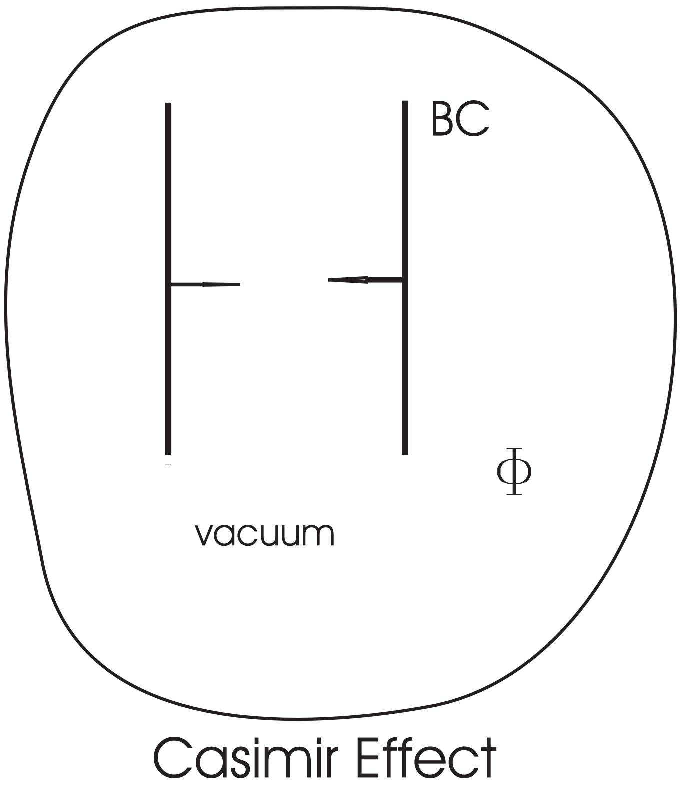

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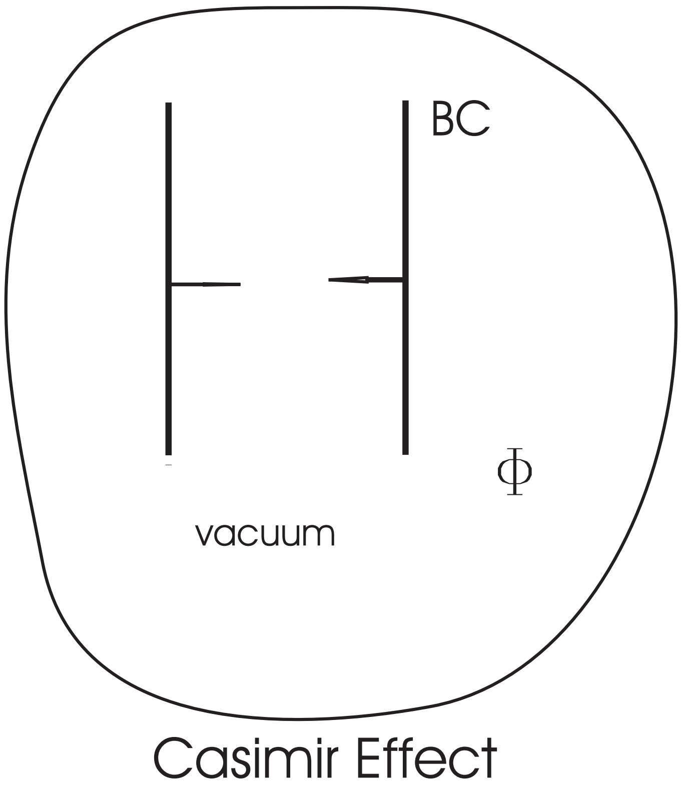

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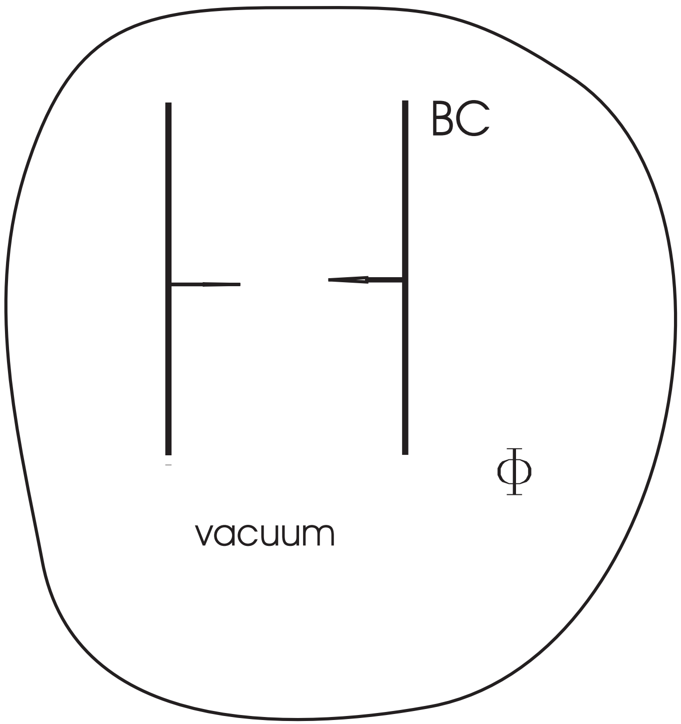
The Casimir Effect



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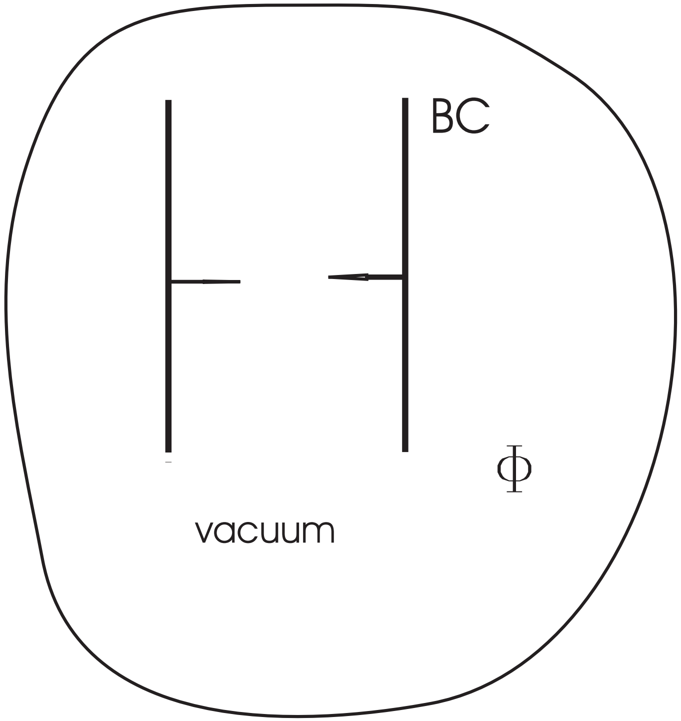

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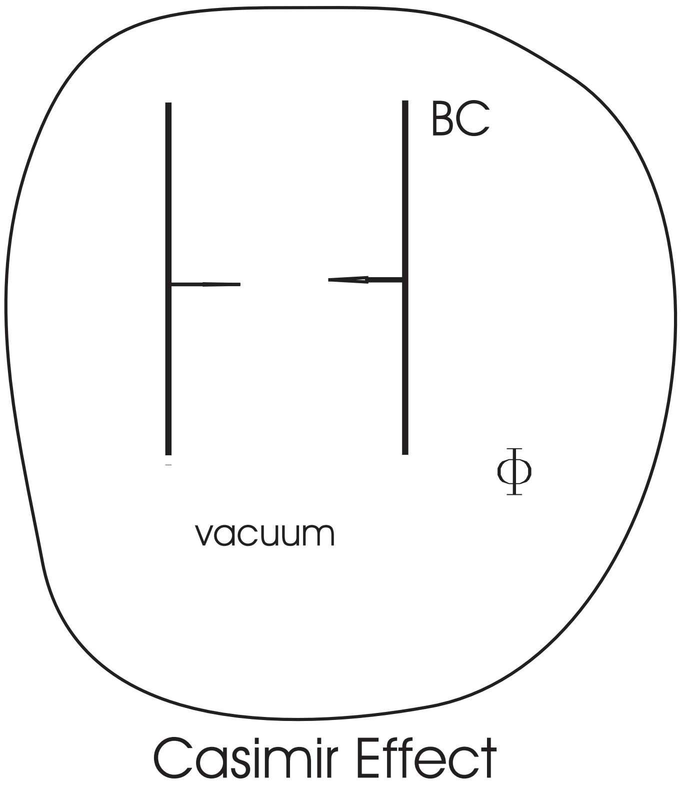

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- Dynamical CE ⇐
- Lateral CE
- Extract energy from vacuum
- CE and the cosmological constant ⇐

The standard approach

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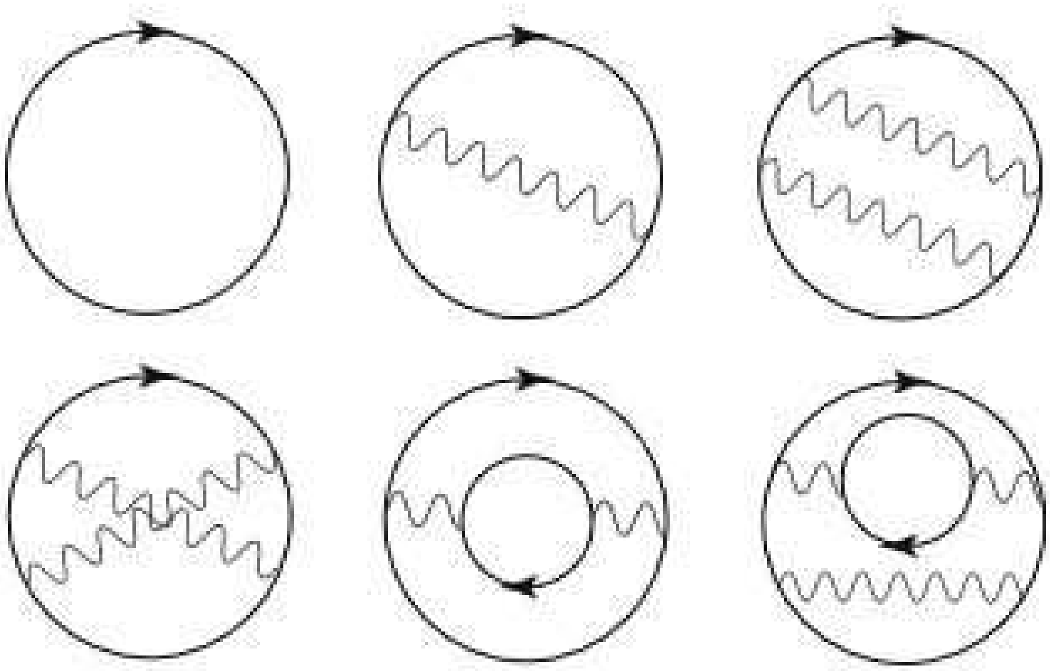

$\Longrightarrow$ Casimir force: calculated by computing change in zero point energy of the em field

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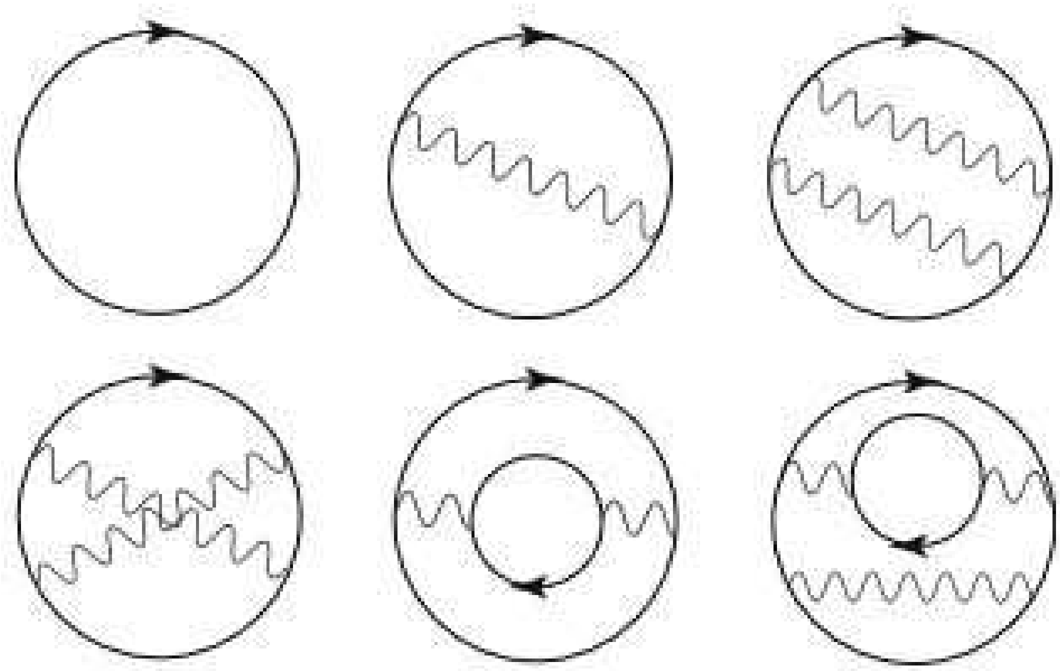

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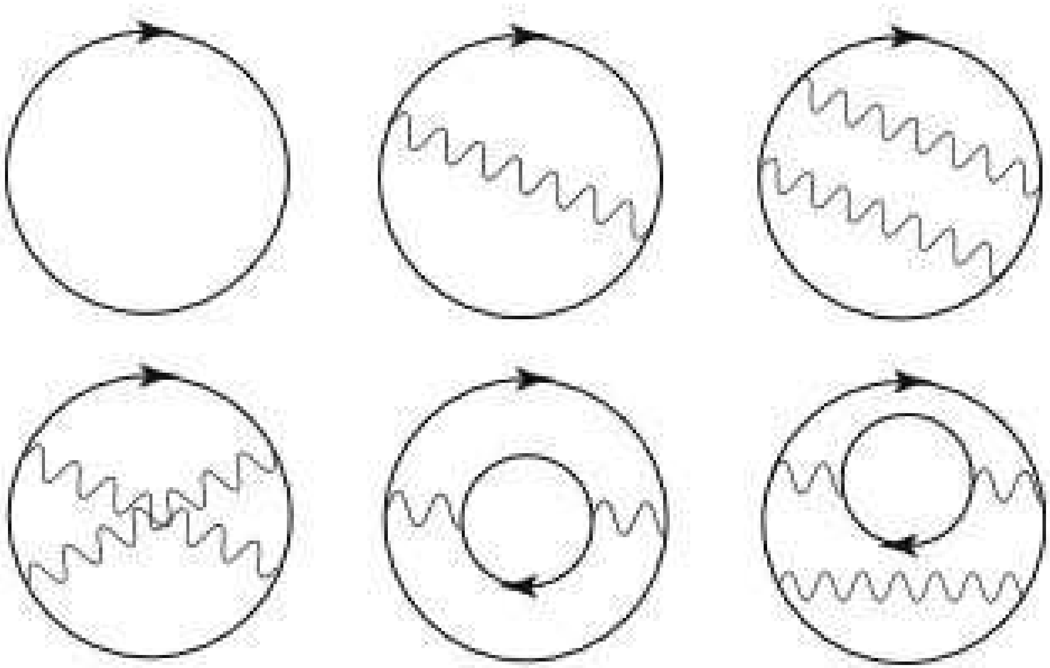

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In modern language the Casimir energy can be expressed in terms of the trace of the Greens function for the fluctuating field in the background of interest (conducting plates)

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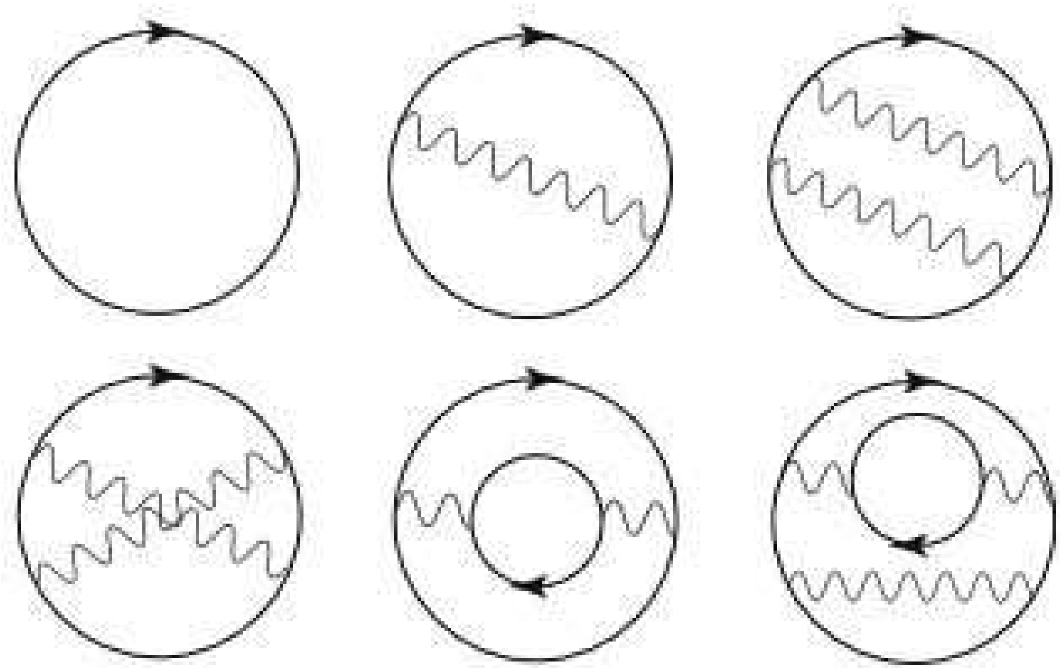

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$\mathcal{G}$ full Greens function for the fluctuating field

$\mathcal{G}_0$ free Greens function — Trace is over spin

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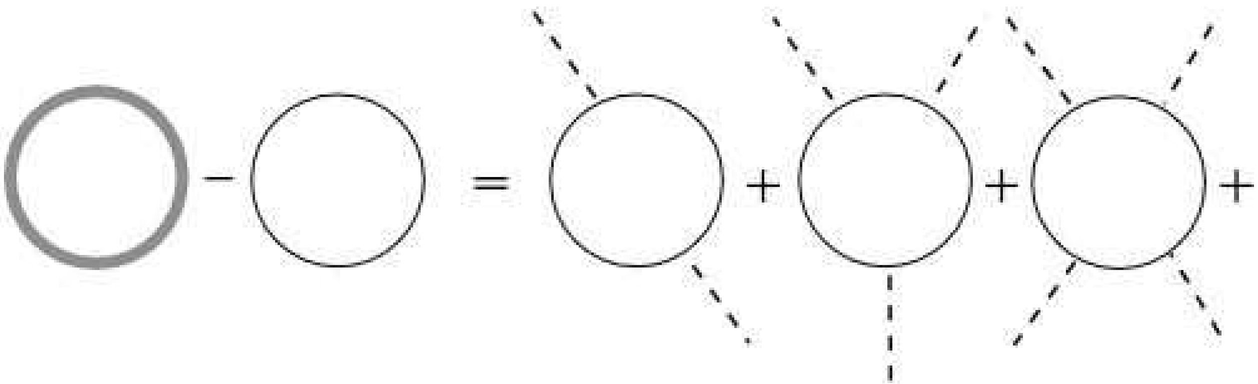

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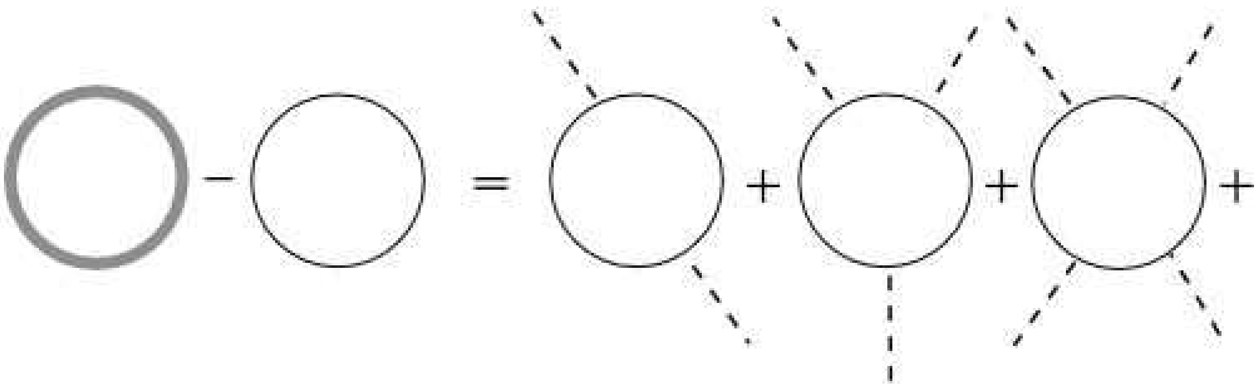

$\Longrightarrow$ "Experimental confirmation of the Casimir effect does not establish the reality of zero point fluctuations" [R. Jaffe et. al.]

The Dynamical Casimir Effect

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- The dissipative part we obtain agrees with the other methods. But those have problems with the reactive part, which in general yields a non-positive energy $\Longrightarrow$ EXPERIMENT

Quantum Vacuum Fluct's & the CC

- The main issue: S.A. Fulling et. al., hep-th/070209v2

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- Idea: zero point fluctuations can contribute to the cosmological constant Ya.B. Zeldovich '68

- Relativistic field: collection of harmonic oscill's (scalar field)

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- What we do consider —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:

  $\Longrightarrow$ kind of cosmological Casimir effect

Cosmo-Topological Casimir Effect

- Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

  * L. Parker & A. Raval, VCDM, vacuum energy density
  * C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two 10^{-2}mm dims, bulk vs brane Susy breaking scales
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  - (c) supergraviton theories (discret dims, deconstr)

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Euler ζ function [Riemann]

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$$\Downarrow$$

Operator ζ functions

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$\swarrow \quad \searrow$

(i) Casimir Eff: NEM, Dyn [EE et al]

(ii) CC, Dark E, accel U [S Hawking]