ICE CSIC IEEC

ζ- REGULARIZATION TECHNIQUES & APPLICATIONS TO COSMOLOGY

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“Quantum Field Theory and Gravity”, Tomsk, July 2, 2007

Outline of this presentation

- Euler and the Zeta Function

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- ΨDOs, Zeta Functions, Determinants, and Traces

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- Quantum vacuum fluctuations and zeta regularization
- The Casimir effect and the cosmological constant
- Other applications in Physics:
  - Non-commutative QFTs (quadratic standard case)
  - Some new developments (quadr non-standard case)

Euler and the Zeta Function

- How did Euler discover the zeta function?
  [R Ayoub, Am Math Month 81 (1974) 1067]. The harmonic series

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

has an infinite sum. Euler: what about the 'prime harmonic series'

$$PH = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$

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- Euler considered $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$

- Provided s is bigger than 1, you can split it up

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- Idea: s closer to 1, first sum increases without bound. Key step is the celebrated equation (p prime)

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- For any p prime and any $s>1$, set $x=1/p^s$ to give:

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- Euler multiplied together these inf sums: his inf product as single inf sum:

$$\frac{1}{p_1^{k_1 s}\cdots p_n^{k_n s}}$$

  $p_1,\ldots,p_n$ primes $k_1,\ldots,k_n$ positive integers, each such combination occurs exactly once, rhs just rearrangement of $\zeta(s)$.

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- Euler's infinite product formula for $\zeta(s)$ marked the beginning of analytic number theory.

- Dirichlet modified the zeta function: primes separated into categories, depending on the remainder when divided by k:

$$L(s,\chi) = \frac{\chi(1)}{1^s} + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \frac{\chi(4)}{4^s} + \cdots$$

where $\chi(n)$ is a special kind of function (Dirichlet 'character') that splits the primes in the required way.

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- Conditions: (i) $\chi(mn) = \chi(m)\chi(n)$ for any m, n
  (ii) $\chi(n) = \chi(n+k), \ \forall n$
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- Any function $L(s,\chi)$, where s real number greater than 1 and χ character, is known as a Dirichlet L-series. Euler zeta function is a special case: $\chi(n) = 1$ for all n. Another ex. $\chi(n) = \mu(n)$ (Möbius)

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- Subsequent generalizations: allow s and $\chi(n)$ to be complex. The celebrated Riemann zeta function, Hurwitz, Epstein, etc. Many results about prime numbers were proven: L-series provide a powerful tool for the study of the primes [Keith Devlin]

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m $\quad$ M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function
$$f \in \mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n \right\}$$
- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

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– – ΨDOs are basic tools both in Mathematics & in Physics – –

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[Fredenhagen, Brunetti, . . . R. Wald '06]

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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$$\zeta_A(s) = \text{tr}\, A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re}\, s > \frac{n}{m} := s_0$$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg}\, \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec}\, A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(d) The only possible singularities of $\zeta_A(s)$ are poles at

$$s_j = (n - j)/m, \qquad j = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

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- This operator is traceclass $\forall t > 0$, and as $t \downarrow 0$

$$\operatorname{tr} e^{-tA} \sim \sum_{j=0}^{\infty} \alpha_j(A) t^{(j-n)/m} + \sum_{k=1}^{\infty} \beta_k(A) t^k \log t$$

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- By Mellin transform:

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tA}\, t^{s-1}\, dt$$

- $\zeta_A(s)$ has a meromorphic extension with only possible poles at

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at most simple at $s_j \notin -\mathbb{N}$, at most double at $s_j \in -\mathbb{N}$

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- If $A \geq 0$ all holds for $A-$ Ker$A$, subtract DimKer to Res at 0
- If $s_j \in \mathbb{N}$, then $\alpha_j(A)$ is not locally computable

P.B. Gilkey & G. Grubb, CPDE, 1998

G. Cognola, L. Vanzo, S. Zerbini, JMP, 1992

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Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \; Re\, s > 1$ (& analytic cont)

Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr } H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \text{tr } e^{-tH}$, Re $s > s_0$

Derivative: $\zeta_H'(0) = -\sum_{i\in I} \ln \lambda_i$

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$$\prod_{i\in I} \lambda_i \quad ?! \qquad\qquad \ln \prod_{i\in I} \lambda_i = \sum_{i\in I} \ln \lambda_i$$

Riemann zeta func: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \; Re\, s > 1$ (& analytic cont)

Definition: zeta function of H $\qquad \zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \text{tr}\; H^{-s}$

As Mellin transform: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1}\, \text{tr}\; e^{-tH}, \; \text{Re}\, s > s_0$

Derivative: $\zeta_H'(0) = -\sum_{i\in I} \ln \lambda_i$

Determinant: Ray & Singer, '67

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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

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- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = {\sum}'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j) \,\text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[\text{PP } \zeta_A(-k) + \psi(k+1) \,\text{Res}_{s=-k} \zeta_A(s)\right], \quad s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$\text{PP } \phi := \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p} \phi(s)}{s-p}\right]$$

The Dixmier Trace

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- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

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$$\text{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

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- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [Connes]

$$\text{Dtr}\, T = \lim_{s\to 1^+} (s-1)\zeta_{T^{-1}}(s)$$

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$$\text{res}\, A = \int_{S^*M} \text{tr}\, a_{-n}(x,\xi)\, d\xi$$

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- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

Singularities of ζ_A

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- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x, \xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1} = 0$:

  $\text{Res}_{s=s_k} \zeta_A(s) = \frac{1}{m} \text{res } A^{-s_k} = \frac{1}{m} \int_{S^*M} \text{tr } a_{-n}^{-s_k}(x, \xi) \, d^{n-1}\xi$

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- Proof. The homog component of degree $-n$ of the corresp power of the principal symbol of A is obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$:

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial \xi^{-1}}\right)^k \left[\xi^{n-k} a^{(k-n)/m}(x,\xi)\right]\Big|_{\xi^{-1}=0} \xi^{-n}$$

Multiplicative or Noncomm Anomaly or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

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$$\delta(A,B) = \frac{\text{res}\left\{[\ln \sigma(A,B)]^2\right\}}{2\ \text{ord}\, A\ \text{ord}\, B\ (\text{ord}\, A + \text{ord}\, B)}$$

where $\quad \sigma(A,B) = A^{\text{ord}\, B} B^{-\text{ord}\, A}$

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi] \ \exp\left\{ - \int d^D x \left[\Phi^\dagger(x) (\quad) \Phi(x) + \cdots \right] \right\}$$

Gaussian integration: $\longrightarrow \quad \det\, (\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A & \\ & B \end{pmatrix}$$

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- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

The Chowla-Selberg Expansion Formula: Basics

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau\in\mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \quad z,t\in\mathbb{C},\ \text{Re}\, t>0$$

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- Higher dimensions: Poisson summ formula (Riemann)

$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p}\tilde{f}(\vec{m})$$

$\tilde{f}$ Fourier transform

[Gelbart + Miller, BAMS '03, Iwaniec, Morgan, ICM '06]

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- Truncated sums $\longrightarrow$ asymptotic series

Extended CS Formulas (ECS)

- Consider the zeta function (Re $s > p/2, A > 0,$ Re $q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A\left(\vec{n}+\vec{c}\right)+q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right)+q\right]^{-s}$$

prime: point $\vec{n}=\vec{0}$ to be excluded from the sum

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- Pole: $s = p/2$ Residue:

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive]

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive]
- Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s}\, \Gamma\left(s - \frac{j}{2}\right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right)\right]$$

[ECS3d]

Zero point energy

QFT vacuum to vacuum transition: $\langle 0|H|0\rangle$

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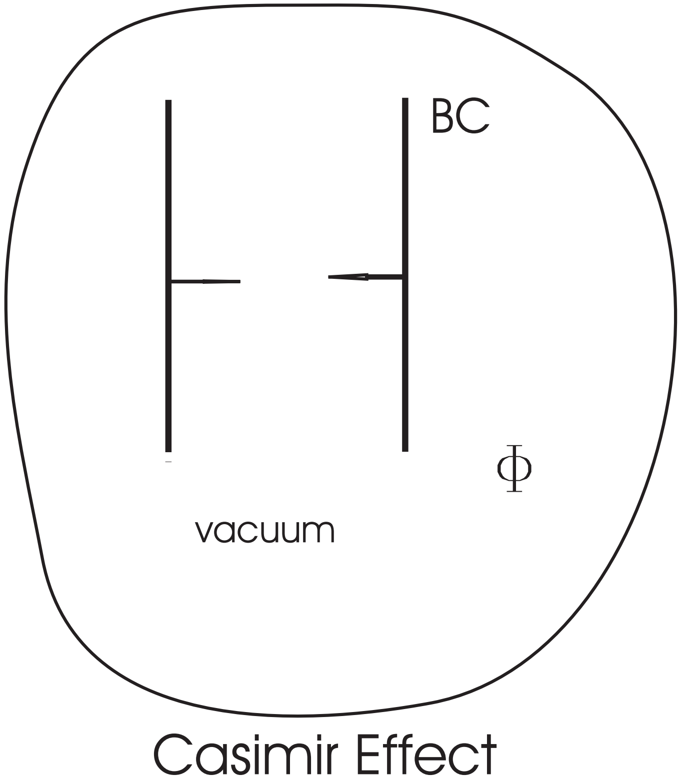
Even then: Has the final value real sense ?

The Casimir Effect

The Casimir Effect

BC e.g. periodic



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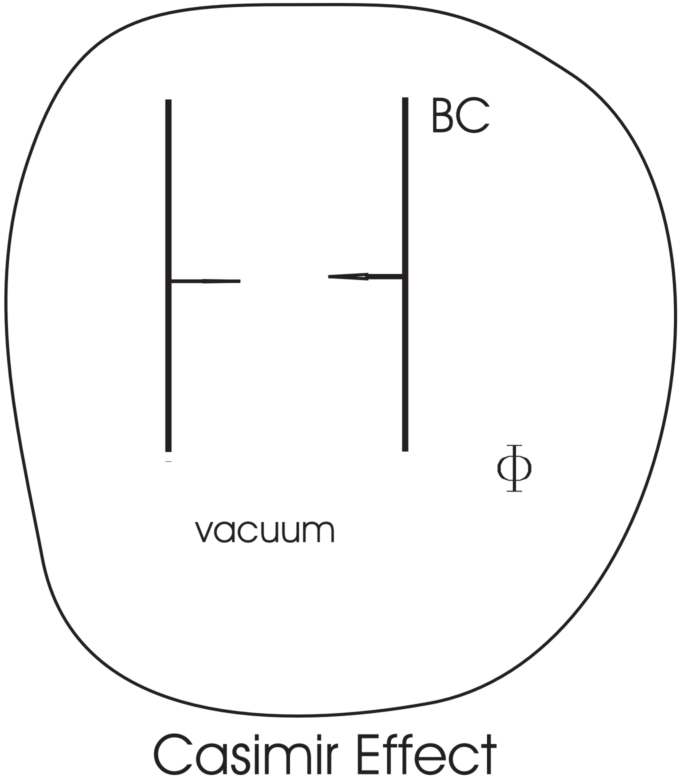

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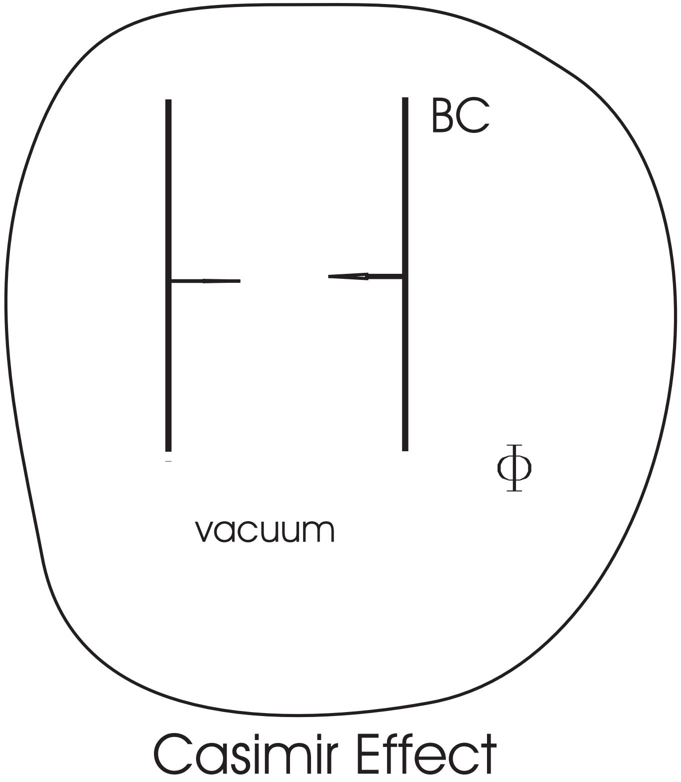

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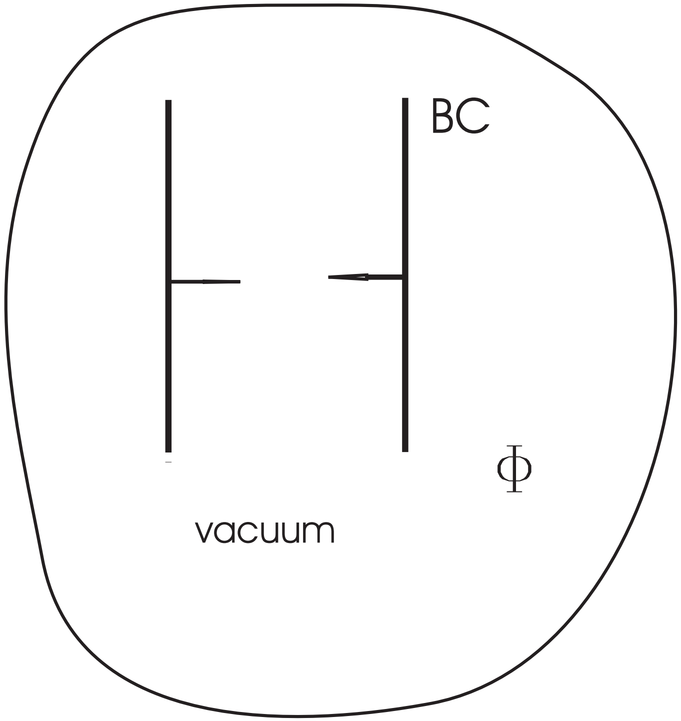

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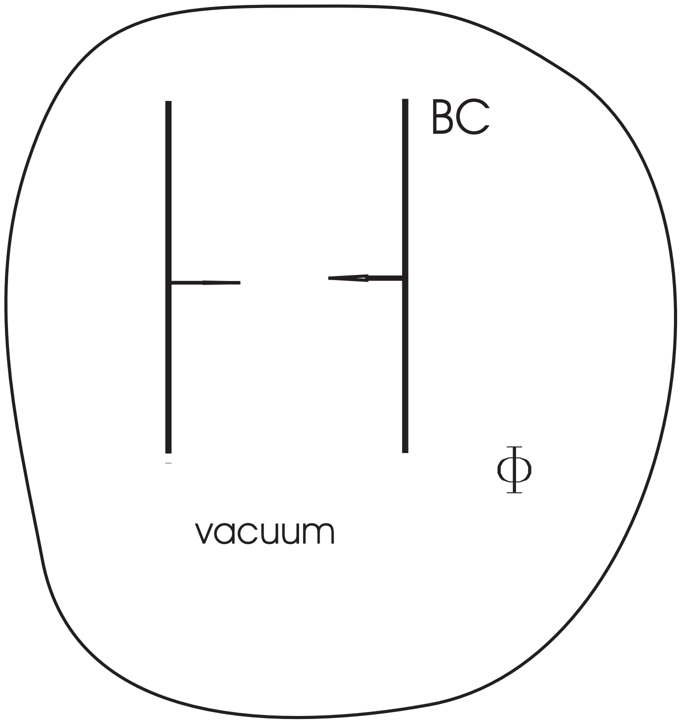
Universal process:

The Casimir Effect



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- Optical cavities
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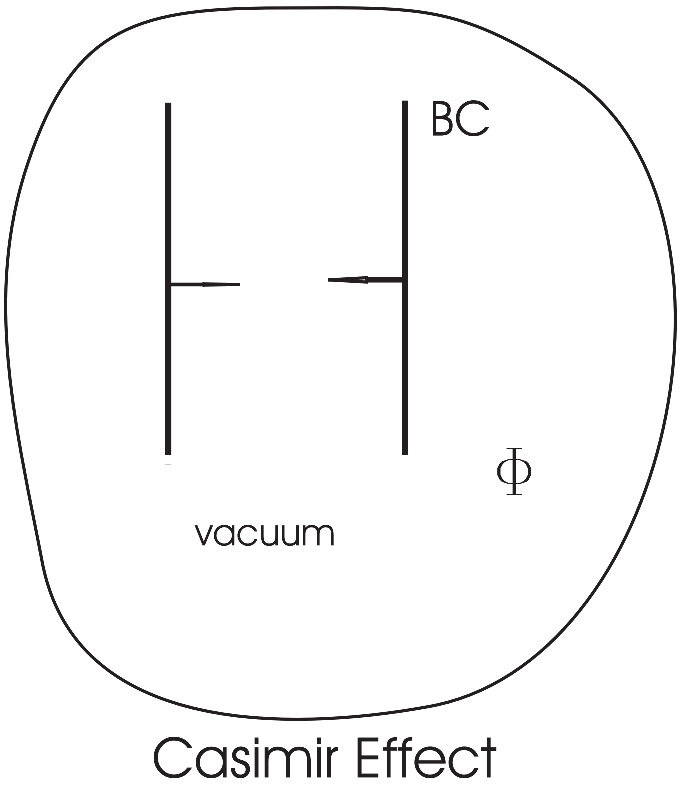

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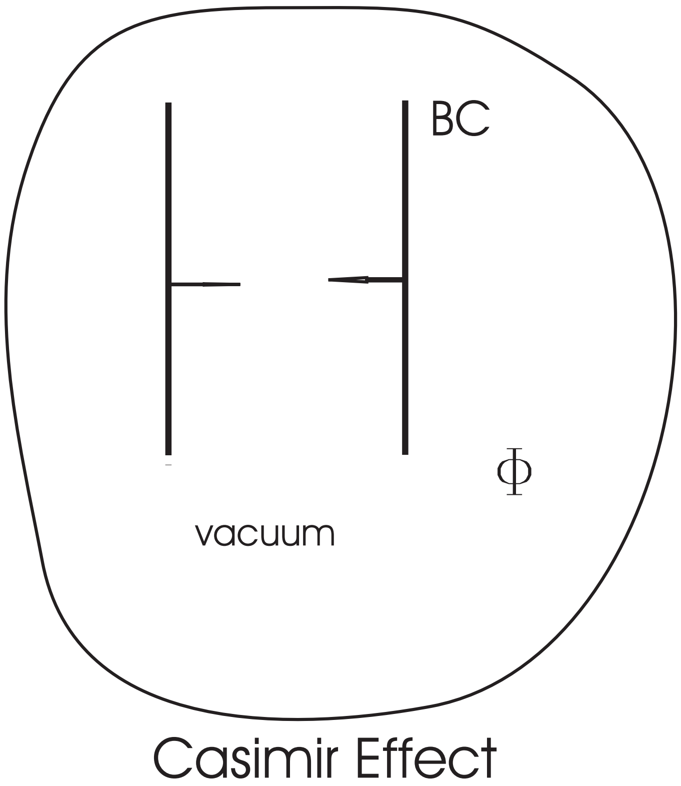

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- Dynamical CE $\Leftarrow$
- Lateral CE
- Extract energy from vacuum
- CE and the cosmological constant $\Leftarrow$

Quantum Vacuum Fluct's & the CC

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  energy ALWAYS gravitates, therefore the energy density of the vacuum, more precisely, the vacuum expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu}$

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- Idea: zero point fluctuations can contribute to the cosmological constant Ya.B. Zeldovich '68

- Relativistic field: collection of harmonic oscill's (scalar field)

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- What we do consider —with relative success in some different approaches— is the additional contribution to the cc coming from the non-trivial topology of space or from specific boundary conditions imposed on braneworld models:

  $\Longrightarrow$ kind of cosmological Casimir effect

Cosmo-Topological Casimir Effect

- Assuming one will be able to prove (in the future) that the ground value of the cc is zero (as many had suspected until recently), we will be left with this incremental value coming from the topology or BCs

  * L. Parker & A. Raval, VCDM, vacuum energy density
  * C.P. Burgess et al., hep-th/0606020 & 0510123: Susy Large Extra Dims (SLED), two 10^{-2}mm dims, bulk vs brane Susy breaking scales
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  - (c) supergraviton theories (discret dims, deconstr)

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- M effective mass term, m arbitrarily small
  (a tiny mass for the field cannot be excluded, and fits well)

* L. Parker & A. Raval, PRL86 749 (2001); PRD62, 083503 (2000)
* V.G. Gurzadyan & S.-S. Xue, Mod Phys Lett A18, 561 (2003)
* A. Chodos & E. Myers, '85-'86

QFT in s-t with non-comm toroidal part

- $D-$dim non-commut manifold: $M = \mathbb{R}^{1,d} \bigotimes \mathbb{T}^p_\theta, \quad D = d + p + 1$

  $\mathbb{T}^p_\theta$ a $p-$dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$

  σ_{jk} a real, nonsingular, antisymmetric matrix of ± 1 entries

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- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s) = \frac{V\,\Gamma(s-(d+1)/2)}{(4\pi)^{(d+1)/2}\,\Gamma(s)} \sum_{\vec{n}\in\mathbb{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s}\left[1+\Lambda\theta^{2-2\alpha}Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s}$$

$\alpha = 2$ bos, $\alpha = 3$ ferm, $V = \text{Vol}\,(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j = a_j^{-1/2}$ compactific radii

- After some calculations,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} \left(-\Lambda\theta^{2-2\alpha}\right)^l \zeta_{Q,\vec{0},0}\left(s+\alpha l-\frac{d+1}{2}\right)$$

for all radii equal to R, with $I(\vec{n}) = \sum_{j=1}^{p} n_j^2$,

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e.g., the Epstein zeta function for the standard quadratic form

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- Classify the different possible cases according to the values of d and $D = d + p + 1$. We obtain, at $s = 0$:

For $d = 2k$ $\begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = 0 \\ \text{if } D = \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \end{cases}$

$$\text{For } d = 2k-1 \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} \begin{Bmatrix} \text{finite,} & \text{for } l \leq k \\ 0, & \text{for } l > k \end{Bmatrix} \Longrightarrow \quad \zeta_\alpha(0) = \text{finite} \\ \\ \text{if } D = 2\alpha l \begin{Bmatrix} \text{pole,} & \text{for } l \leq k \\ \text{finite,} & \text{for } l > k \end{Bmatrix} \Longrightarrow \quad \zeta_\alpha(0) = \text{pole} \end{cases}$$

– Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means multiple of 2α)

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$\Longrightarrow$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$, & specific pole structure

$$\zeta_\alpha(s) = \frac{2^{s-d}\,V}{(2\pi)^{(d+1)/2}\Gamma(s)}\sum_{l=0}^{\infty}\frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)}(-2^\alpha\Lambda\theta^{2-2\alpha})^l\sum_{j=0}^{p-1}(\det A_j)^{-\frac{1}{2}}$$

$$\times\Bigg[\pi^{j/2}a_{p-j}^{-s-\alpha l+(d+j+1)/2}\Gamma(s+\alpha l-(d+j+1)/2)\zeta_R(2s+2\alpha l-d-j-1)$$

$$+\,4\pi^{s+\alpha l-(d+1)/2}a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4}\sum_{n=1}^{\infty}\sum_{\vec{m}_j\in\mathbb{Z}^j}{}'\, n^{(d+j+1)/2-s-\alpha l}$$

$$\times\left(\vec{m}_j^t A_j^{-1}\vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l}\left(2\pi n\sqrt{a_{p-j}\vec{m}_j^t A_j^{-1}\vec{m}_j}\right)\Bigg]$$

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p \ D	even	odd
odd	(1a) pole / finite ($l \geq l_1$)	(2a) pole / pole
even	(1b) double pole / pole ($l \geq l_1, l_2$)	(2b) pole / double pole ($l \geq l_2$)

– General pole structure of $\zeta_\alpha(s)$, for the possible values of D and p being odd or even. Magenta, type of behavior corresponding to lower values of l; behavior in blue corresponds to larger values of l

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- We produce an exact analysis of the one-loop effective action in the 4D, alternative model [EE-Minamitsuji-Naylor, PRD 2007]

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- Accurate evaluation of A_4 is crucial for making physical predictions: hierarchy & CC problems

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where we redefine the terms:

$$\hat{\alpha}(n)=\alpha+\frac{bn}{2a}=\frac{1}{2}+\frac{(1+r)|n|}{2\kappa}, \qquad \hat{q}(n)=-\frac{n^2}{\kappa^2}(1-r)^2-1$$

Extended binomial expansion

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[a(m+\alpha)^2+b(m+\alpha)n+cn^2+q\right]^{-s+1}$$

$$=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[a(m+\hat{\alpha})^2+\hat{q}\right]^{-s+1}$$

$$=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(j+s-1)}{\Gamma(s-1)\,j!}\left[a(m+\hat{\alpha})^2\right]^{1-s-j}\,\hat{q}^j$$

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$$\zeta(s)\big|_{n\neq 0}=\frac{g^{2(1-s)}}{\pi}\int d^2x\sum_{j=0}^{\infty}G(j,s)\sum_{n=1}^{\infty}\left[\frac{n^2}{\kappa^2}(1-r)^2+1\right]^j\zeta_H\left(2s+2j-2,\frac{1}{2}+\frac{1+r}{2\kappa}n\right)$$

$$G(j,s)\equiv\frac{2^{-2(j+s-1)}\Gamma(s+j-1)}{\Gamma(s)\;j!}$$

Analytic continuation of the ζ function

$$P(s) = \frac{g^{2(1-s)}}{\pi} \int d^2x \sum_{j=0}^{\infty} G(j,s) \sum_{n=1}^{\infty} \Big[\Big(\frac{n^2}{\kappa^2} (1-r)^2 + 1 \Big)^j \times \zeta_H \Big(2s+2j-2, \frac{1}{2} + \frac{1+r}{2\kappa} n \Big) - F(n,j;s) \Big]$$

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$\Longrightarrow$ Extend our analysis to the cases of

(1) 6 dimensions

(2) a bulk scalar field with self-interactions and other fields in the multiplets appearing in the supergravity model

Summary:

Euler ζ function [Riemann]

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Operator ζ functions

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(i) Casimir Eff: NEM, Dyn — [EE et al]

(ii) CC, Dark E, accel U — [S Hawking]